H2 Further Mathematics Contents

Matrices

- Augmented matrix of a linear system: $\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}$.
- Gaussian elimination: (Reduced) row-echelon form.
- Trace: Sum of all entries on the main diagonal.
- $(\mathrm{AB})^{\mathrm{T}} = \mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}$
- $AI = IA = A$
- If $\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- $(\mathrm{AB})^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$
- Finding inverse: $(\mathrm{A}|\mathrm{I}) \to (\mathrm{I}|\mathrm{A}^{-1})$
- If $\det(\mathrm{A}) \neq 0$, A is invertible.
- If A is invertible, the equation $\mathrm{A}\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = \mathrm{A}^{-1}\mathbf{b}$; $\mathrm{A}\mathbf{x} = 0$ has only the trivial solution.
- Adjoint of A (adj(A)) is the transpose of its cofactor matrix.
- $\mathrm{A}[\mathrm{adj}(\mathrm{A})] = \det(\mathrm{A})\mathrm{I}$; $\mathrm{A}^{-1} = \frac{1}{\det(\mathrm{A})}\mathrm{adj}(\mathrm{A})$;
- ~~Cramer's Rule: If A is invertible, the solution of $\mathrm{A}\mathbf{x} = \mathbf{b}$ is $x_i = \frac{\det(\mathrm{A}_i)}{\det(\mathrm{A})}$, where A_i is obtained by replacing the i-th column of A by **b**.~~
- Eigenvector of A: $\mathrm{A}\mathbf{x} = \lambda\mathbf{x}$, where λ is the corresponding eigenvalue.
- $\det(\lambda\mathrm{I} - \mathrm{A}) = 0$
- $\mathrm{A} = \mathrm{PDP}^{-1}$, P is matrix of eigenvector and D is matrix of eigenvalue.

Conics

	Ellipse	Hyperbola	Parabola
Geometric Definition	$\mathrm{PF}_1 + \mathrm{PF}_2 = k$	$\lvert\mathrm{PF}_1 - \mathrm{PF}_2\rvert = k$	$\mathrm{PF} = \lvert x_P - x_c\rvert$
Cartesian Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$y^2 = 4cx$
Parametric Equation	$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$	$\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$	$\begin{cases} x = c \\ y = 2c^2 \end{cases}$
Polar Equation $r = \frac{ep}{1+e\cos\theta}$	$0 \le e < 1$	$e > 1$	$e = 1$
Focus	$c^2 = a^2 - b^2$	$c^2 = a^2 + b^2$	c

- DEF definition: The ratio of the distance between P and a fixed point F to the distance between P and a fixed line d: $x = k$ is a fixed value e. $\frac{\sqrt{(x-c)^2+y^2}}{|x-k|} = e$. Let $c - ke^2 = 0$.

<table>
<tr><td></td><td>
<table>
<tr><td>Eccentricity Range</td><td>$0 \le e < 1$</td><td>$e > 1$</td><td>$e = 1$</td></tr>
<tr><td>Eccentricity $e = \frac{c}{a}$</td><td>$e = \sqrt{1 - \frac{b^2}{a^2}}$</td><td>$e = \sqrt{1 + \frac{b^2}{a^2}}$</td><td>$e = \frac{c}{a}$</td></tr>
</table>
</td></tr>
<tr><td>Mathematical Induction</td><td>
• Let P_n be the statement …. When $n = 1$, LHS=RHS, hence P_1 is true; Assume that P_k is true for some value of k, prove P_{k+1} is true. Since P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in Z^+$.
</td></tr>
<tr><td>Linear Spaces</td><td>
• S is a basis of P if S is linearly independent and S spans P.
• Dimension is the number of vectors in the basis of V.
• Null space of A is the set of all the solutions of $A\mathbf{x} = \mathbf{0}$.
• Rank of A (rank(A)) is the dimension of row space of A.
• Nullity of A (nullity(A)) is the dimension of null space of A.
• For an $m \times n$ matrix A, $\text{rank(A)} + \text{nullity(A)} = n$.
• Linear transformation: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(k\mathbf{u}) = kT(\mathbf{u})$.
• For $T: R^n \to R^n$, $\text{rank(T)} + \text{nullity(T)} = n$.
</td></tr>
<tr><td>Recurrence Relation</td><td>
• First-order recurrence relation: $x_{n+1} = ax_n + b$
$$x_{n+1} = ax_n + b = a(ax_{n-1} + b) + b = a^2x_{n-1} + (b + ab)$$
$$= \cdots = a^{n+1}x_0 + b\frac{1 - a^{n+1}}{1 - a}$$
• Second-order recurrence relation: $ax_{n+2} + bx_{n+1} + cx_n = 0$ has a general solution of $x_n = Am_1^n + Bm_2^n$. When $m_1 = m_2$, the solution is $x_n = (An + B)m^n$.
</td></tr>
<tr><td>Numerical Methods</td><td>
• Interval bisection: sign of $f(\frac{a+b}{2})$.
• Linear interpolation: sign of $f(\frac{a|f(a)|+b|f(b)|}{|f(a)|+|f(b)|})$.
• Newton-Raphson: $x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$
• Recurrence relation: $x_{r+1} = f(x_n)$
• Trapezium rule: $\int_a^b f(x)\,dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$
• Simpson's rule: $\int_a^b f(x)\,dx \approx \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{2n-1}) + f(x_{2n})]$
</td></tr>
<tr><td>Further Application of Integration</td><td>
• Area under polar curve: $\int \frac{1}{2} r^2\,d\theta$ (area of sectors of circle)
• Volume (disc): $\int \pi y^2\,dx$ (volume of cylinder)
• Volume (shell): $\int 2\pi yx\,dy$ (perimeter×thickness)
</td></tr>
</table>

<table>
<tr>
<td></td>
<td>

- Arc length: $\int \sqrt{dx^2 + dy^2}$ (Pythagoras Theorem)
- Arc length of polar curve: $\int \sqrt{dr^2 + (rd\theta)^2}$
- Surface area: $\int 2\pi y\sqrt{dx^2 + dy^2}$ (perimeter×thickness)

</td>
</tr>
<tr>
<td>Further Differential Equation</td>
<td>

- Integration factor: For $\frac{dy}{dx} + P(x)y = Q(x)$, multiply both sides by $I(x) = e^{\int P(x)dx}$:

$$I(x)\frac{dy}{dx} + I(x)P(x)y = Q(x)I(x)$$
$$I(x)\frac{dy}{dx} + I'(x)y = Q(x)I(x)$$
$$\frac{d}{dx}(I(x)y) = Q(x)I(x)$$
$$y = \frac{\int Q(x)I(x)dx}{I(x)}$$

- Solution to second-order differential equation: $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + c = 0$

$\Delta > 0$	$y = Ae^{m_1x} + Be^{m_2x}$
$\Delta = 0$	$y = (A + Bx)e^{mx}$
$\Delta < 0$	$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

- Euler's method: $y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n)$
- Improved Euler's method: $\begin{cases} y^*_{n+1} = y_n + \Delta t \cdot f(t_n, y_n) \\ y_{n+1} = y_n + \Delta t \cdot \left[\frac{f(t_n,y_n)+f(t_{n+1},y^*_{n+1})}{2}\right] \end{cases}$

</td>
</tr>
<tr>
<td>Further Complex Number</td>
<td>

- de Moivre's Theorem: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$
- Roots of $z^n = re^{i\theta}$: $z = r^{\frac{1}{n}}e^{i\frac{\theta+2k\pi}{n}}$

</td>
</tr>
<tr>
<td>Further Special Discrete Probability Distributions</td>
<td>

- Conditions for Poisson Distribution: Independence, constant mean rate, randomness
- If $X \sim Po(\lambda)$, $E(X) = \lambda$, $Var(X) = \lambda$
- Conditions for geometric distribution: Independence, two outcomes, constant probability of success
- Memoryless property: If $X \sim Geo(p)$, $P(X > m + n \mid X > n) = P(X > m)$

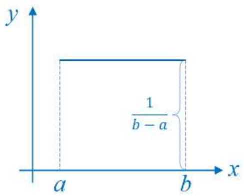
</td>
</tr>
</table>

<table>
<tr><td>Continuous Random Variables</td><td>

$\int_{-\infty}^{+\infty} f(x)dx = 1$
Cumulative density function: $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt$
Median, m: $F(m) = 1$
Mode, m^*: $f(m^*) \ge f(x)$ for all x
$E(X) = \int_{-\infty}^{+\infty} xf(x)dx$
$E(g(X)) = \int_{-\infty}^{+\infty} g(x)f(x)dx$
$P(X \le x) = P(Y \le f(x))$

</td></tr>
<tr><td>Special Continuous Probability Distributions</td><td>

Uniform distribution: $E(X) = \frac{a+b}{2}$, $Var(X) = \frac{1}{12}(b-a)^2$



</td></tr>
<tr><td>Confidence Intervals</td><td>

Unbiased estimator for μ: $\bar{X}$
Unbiased estimator for σ^2: $S^2 = \frac{n}{n-1}{\sigma_n}^2$
Unbiased estimator for p: $P_S = \frac{X}{n}$
Explain in context a confidence interval:
If a large number of samples are taken randomly, and their (1-α) confidence intervals for μ are found, then we say that (1-α) of such intervals will contain μ.
Confidence interval for population mean:
<table>
<tr><td>Case I:
σ^2 is known</td><td>A $(1-\alpha)\times 100\%$ confidence interval:
$(\bar{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}})$</td></tr>
<tr><td>Case II:
σ^2 is unknown, large sample size</td><td>A $(1-\alpha)\times 100\%$ confidence interval:
$(\bar{X} - z_{1-\frac{\alpha}{2}}\frac{s}{\sqrt{n}}, \bar{X} + z_{1-\frac{\alpha}{2}}\frac{s}{\sqrt{n}})$</td></tr>
<tr><td>Case III:
σ^2 is unknown (normal)</td><td>A $(1-\alpha)\times 100\%$ confidence interval:
$(\bar{X} - t_{n-1,1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X} + t_{n-1,1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}})$</td></tr>
</table>

Confidence interval for population proportion:
$$(P_s - z_{1-\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}}, P_s + z_{1-\frac{\alpha}{2}}\sqrt{\frac{p(1-p)}{n}})$$

</td></tr>
<tr><td>Further Hypothesis</td><td>

t-test is used when population is normally distributed and sample size is small.

</td></tr>
</table>

Testing	• Presentation: Test H_0 against H_1 (state both hypotheses). Level of significance: α% (lower/upper/two-tailed). Under H_0, $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$, hence $Z = \frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \sim N(0, 1)$ <u>Method 1</u> Critical region: $z < a$ (taking lower-tailed as an example). Observed test statistic: $z = \frac{\bar{x}-\mu}{\sqrt{\sigma^2/n}} > a$ Hence we do not reject H_0. <u>Method 2</u> By using GC, *p*-value = *p* Since $p >$ α%, we do not reject H_0. Therefore, there is insufficient evidence at α% level of significance to claim H_1 (stated). • When using CLT, remember to convert sample variance to unbiased estimate of population variance. • Assumptions for *t*-test: Normal distribution, randomness, independence. • Value of population mean specified by the null hypothesis is contained in the α% confidence interval. ↔ We do not reject the null hypothesis at a $(1-\alpha\%)$ level of significance. • Two samples test: Under H_0, $Z = \frac{(\overline{X_1}-\overline{X_2})-(\mu_1-\mu_2)}{\sqrt{\frac{{\sigma_1}^2}{n_1}+\frac{{\sigma_2}^2}{n_2}}} \sim N(0, 1)$ • For two samples test, X_1 and X_2 are drawn independently. • Unbiased estimate for common variance: $s = \frac{n_1{\sigma_{s1}}^2+n_2{\sigma_{s2}}^2}{n_1+n_2-2}$ • *t*-test is used when population is normally distributed and at least one sample size is small. • Paired sample *t*-test is used when each subject is measured twice (e.g. before and after tutorials). In this case, under H_0, $T = \frac{\bar{D}-d_0}{s_D/\sqrt{n}} \sim t(n-1)$ • For paired-sample test, the difference is normally distributed. • Pearson's statistic: $\chi^2 = \sum_{i=1}^{n}\frac{(O_i-E_i)^2}{E_i} \sim \chi^2(v)$, provided that $E_i \geq 5$. • Degree of freedom, v = number of classes – number of estimated parameters – 1 for goodness of fit. • Collapse whenever $E_i < 5$. • Degree of freedom, v = (no. of rows – 1)×(no. of columns – 1) for independence.

Non-parametric Tests	• Non-parametric tests test the median. The null hypotheses are to claim that the median difference is 0. • Continuity correction: $P(X = k) \Longrightarrow P(k - \frac{1}{2} < X < k + \frac{1}{2})$; $P(X < k) \Longrightarrow P(X < k + \frac{1}{2})$