CS1231 DISCRETE STRUCTURES Cheatsheet

Properties of Tian Xiao

Logical Statements

Basic Operators: and ($\wedge$), or ($\vee$), not ($\sim$)

Laws of Logical Equivalence

01. Commutative Laws: $p \vee\!\!\!\!\wedge q \equiv q \vee\!\!\!\!\wedge p$
02. Associative Laws: $(p \vee\!\!\!\!\wedge q) \vee\!\!\!\!\wedge r \equiv p \vee\!\!\!\!\wedge (q \vee\!\!\!\!\wedge r)$
03. Distribution Laws: $p \wedge (q \vee r)$; $p \vee (q \wedge r)$
04. Identity Laws: $p \wedge$ **True** $\equiv p$; $p \vee$ **False** $\equiv p$
05. Negation Laws: $p \wedge \sim p \equiv$ **False;** $p \vee \sim p \equiv$ **True**
06. Double Negative Laws: $\sim(\sim p) \equiv p$
07. Idempotent Laws: $p \wedge p \equiv p$; $p \vee p \equiv p$
08. Universal Bound Laws: $p \wedge$ **False** $\equiv$ **False**; $p \vee$ **True** $\equiv$ **True**
09. De Morgan's Laws: $\sim(p \wedge q) \equiv \sim p \vee \sim q$; $\sim(p \vee q) \equiv \sim p \wedge \sim q$
10. Absorption Laws: $p \vee (p \wedge q) \equiv p$; $p \wedge (p \vee q) \equiv p$
11. Negation of True/False: ~**True** $\equiv$ **False**; ~**False** $\equiv$ **True**

Conditional Statements

01. Implication Law: $p \to q \equiv \sim p \vee q$
02. Contrapositive: $\sim q \to \sim p \equiv p \to q$
03. Converse: $q \to p$
04. Inverse: $\sim p \to \sim q$

Rules of Inference

Modus Ponens

$p \to q$
p
$\bullet\ q$

Modus Tollens

$p \to q$
$\sim q$
$\bullet \sim p$

Generalisation

p
$\bullet\ p \vee q$

Conjunction

p
q
$\bullet\ p \wedge q$

Elimination

$p \vee q$
$\sim q$
$\bullet\ p$

Specialisation

$p \wedge q$
$\bullet\ p$

Transitivity

$p \to q$
$q \to r$
$\bullet\ p \to r$

Division into Cases

$p \vee q$
$p \to r$
$q \to r$
$\bullet\ r$

Contradiction

$\sim p \to$ **False**
$\bullet\ p$

Quantitative Operators: $\exists!$ (there exists one and only one)

Quantitative Statements

01. Negation: $\forall \to \exists$; $P(x) \to \sim P(x)$
02. Contrapositive: $\forall x \in D, \sim Q(x) \to \sim P(x)$
03. Converse: $\forall x \in D, Q(x) \to P(x)$
04. Inverse: $\forall x \in D, \sim P(x) \to \sim Q(x)$

Universal Instantiation

$\forall x \in D, P(x)$
$a \in D$
$\bullet\ P(a)$

Rule + Universal Instantiation
= Universal Rule

Definition of Numbers

01. Even: $\exists k \in \mathbf{Z}$ such that $x = 2k$
02. Odd: $\exists k \in \mathbf{Z}$ such that $x = 2k + 1$
03. Prime: $\forall r, s \in \mathbf{Z}+$, $n = rs \Rightarrow (r = 1, s = n)$ or $(r = n, s = 1)$
04. Composite: $\exists r, s \in \mathbf{Z}+$, $(n = rs)$ and $(1 < r, s < n)$
05. Rational: $\exists p, q \in \mathbf{Z}$, $r = \frac{p}{q}$ and $q \neq 0$
06. Divisible: $\exists k \in \mathbf{Z}$, $n = dk$

Proof by Contradiction

The contrapositive of $P(x) \to Q(x)$ is $\sim Q(x) \to \sim P(x)$.

1. Prove the contrapositive statement through a direct proof.
   1.1. Suppose $x \in \mathbf{D}$ such that $Q(x)$ is **False**.
   1.2. Show that $P(x)$ is **False**.
2. Therefore, the original statement $P(x) \to Q(x)$ is **True**.

Sets and Functions

Set Concepts

01. Equal Sets: $A = B \Leftrightarrow x \in A \leftrightarrow x \in B$
02. Subset: $A \subseteq B \Leftrightarrow \forall x,\ x \in A \to x \in B$
03. Finite Set: $|S| = n$, where n is called cardinality
04. Power Set: $P(A)$ is the set of all subsets of A
05. Cartesian Product: $A \times B = \{(a, b) \mid a \in A, b \in B\}$

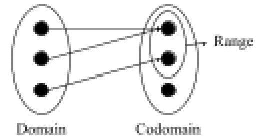
Set Operations

01. Union: $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$
02. Intersection: $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$
03. Compliment: $B - A = B \backslash A = B \cap \overline{A}$
04. Complement: $\overline{A} = U - A$
05. Disjoint: $A \cap B = \emptyset$

Set Identities

01. Commutative Laws: $A \cup\!\!\!\!\cap B \equiv B \cup\!\!\!\!\cap A$
02. Associative Laws: $(A \cup\!\!\!\!\cap B) \cup\!\!\!\!\cap C \equiv A \cup\!\!\!\!\cap (B \cup\!\!\!\!\cap C)$
03. Distribution Laws: $A \cap (B \cup C)$; $A \cup (B \cap C)$
04. Identity Laws: $A \cap U = A$; $A \cup \emptyset = A$
05. Negation Laws: $A \cap \overline{A} = \emptyset$; $A \cup \overline{A} = U$
06. Double Negative Laws: $\overline{(\overline{A})} = A$
07. Idempotent Laws: $A \cap A = A$; $A \cup A = A$
08. Universal Bound Laws: $A \cap \emptyset = \emptyset$; $A \cup U = U$
09. De Morgan's Laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$; $\overline{A \cup B} = \overline{A} \cap \overline{B}$
10. Absorption Laws: $A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$
11. Negation of True/False: $\overline{U} = \emptyset$; $\overline{\emptyset} = U$

Function Concepts



01. $f : X \to Y$ is injective iff $\forall a, b \in X, f(a) = f(b) \Rightarrow a = b$
02. $f : X \to Y$ is surjective iff $\forall y \in Y, \forall x \in X\ (f(x) = y)$
03. Bijective: 1-1 + onto
04. Inverse Functions: Let $f : X \to Y$ be a bijection. Then its inverse $g : Y \to X$: $\forall y \in Y,\ g(y) = x \Leftrightarrow f(x) = y$
05. Image: $f(X) = \{f(x) \mid x \in X\}$
06. Preimage: $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\}$

Induction

Mathematical Induction

1. For each $n \in D$, let $P(n)$ be the proposition <XXX>.
2. (Base step) $P(1)$ is true because <RRR>.
3. (Induction step)
   3.1. Let $k \in D$ such that $P(k)$ is true, i.e. <XXX>.
   3.2. <YYY>
   …
   3.n. Thus $P(k + 1)$ is true.
4. Hence $\forall n \in D,\ P(n)$ is true by Mathematical Induction.

Strong Induction

1. For each $n \in D$, let $P(n)$ be the proposition <XXX>.
2. (Base step) $P(1)$ is true because <RRR>.
3. (Induction step)
   3.1. Let $k \in D$ such that $P(1), \ldots, P(k)$ is true, i.e. <XXX>.
   3.2. <YYY>
   …
   3.n. Thus $P(k + 1)$ is true.
4. Hence $\forall n \in D,\ P(n)$ is true by Strong Induction.