MA1101R Linear Algebra

Linear Systems and Gaussian Elimination

- Linear Equation in n-variables: $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$.
- Linear System: A finite set of linear equations.
- Inconsistent Linear System: It has no solution.
- Augmented Matrix: $\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & b_n \end{array}\right]$.
- Elementary Row Operations (Row Equivalent):
  1. Multiply a row by a non-zero constant (cR_i).
  2. Interchange two rows ($R_i \leftrightarrow R_j$).
  3. Add a multiple of one row to another row ($R_i + cR_j$).
- Row-Echelon Forms (Gaussian Elimination):
  1. Rows of entirely zeros are grouped at the bottom.
  2. Lower leading entries (pivot points) are to the right of upper ones.
- Reduced Row-Echelon Forms (Gauss-Jordan Elimination):
  1. Leading entries of non-zero rows are 1.
  2. In each pivot columns, all entries except pivot points are 0.
- Number of Solutions:
  1. No solution, if last column of RREF of augmented matrix is a pivot column.
  2. One solution, if only last column of RREF is not a pivot column.
  3. Infinite solution, if not only last column of RREF is not a pivot column.
- Homogenous Linear System: $\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \end{array}\right]$.
- A homogenous linear system has either trivial solution or infinitely many solutions.

Matrices

- Diagonal Matrix: Only the diagonal entries are non-zero.
- Scalar Matrix: A diagonal matrix where all the diagonal entries are equal.
- Identity Matrix (*I*): A diagonal matrix where all the diagonal entries are 1.
- Zero matrix: All the entries are 0.
- Symmetric Matrix: $a_{i,j} = a_{j,i}$.
- Triangular Matrix:
  1. Upper triangular matrix: $a_{i,j} = 0$ whenever $i > j$.
  2. Lower triangular matrix: $a_{i,j} = 0$ whenever $i < j$.
- Properties of Matrix Operations:
  1. $A + B = B + A$.
  2. $A + (B + C) = (A + B) + C$.
  3. $c(A + B) = cA + cB$.
  4. $(c + d)A = cA + dA$.
  5. $(cd)A = c(dA) = d(cA)$.
  6. $A + 0 = 0 + A = A$.
  7. $A - A = 0$.
  8. $0A = 0$.
  9. $A(BC) = (AB)C$.
  10. $A(B + C) = AB + AC$.

11. $(A+B)C = AC + BC$.
12. $c(AB) = A(cB) = (cA)B$.
13. $A0 = 0$.
14. $AI = A$.

- Pre-multiplication of A to B: AB.
- Post-multiplication of A to B: BA.
- Transpose: $A_{i,j}^T = A_{j,i}$.
  1. $(A^T)^T = A$.
  2. $(A+B)^T = A^T + B^T$.
  3. $(cA)^T = c(A^T)$.
  4. $(AB)^T = B^T A^T$.
- Inverse: $AA^{-1} = I$ (invertible, non-singular, unique).
  1. $(cA)^{-1} = \frac{1}{c} A^{-1}$.
  2. ${A^T}^{-1} = {A^{-1}}^T$.
  3. ${A^{-1}}^{-1} = A$.
  4. $(AB)^{-1} = B^{-1}A^{-1}$.
- Cancellation Law: If A is invertible, then if $AB_1 = AB_2$ or $B_1A = B_2A$, $B_1 = B_2$.
- Elementary Matrices: Elementary row operations to the identity matrix.
- Determinant: Calculated from cofactor expansion along any row/column.
  1. $cR_i \rightarrow c\det(A)$.
  2. $R_i \leftrightarrow R_j \rightarrow -\det(A)$.
  3. $R_i + cR_j \rightarrow \det(A)$.
  4. $\det(A^T) = \det(A)$.
  5. $\det(cA) = c\det(A)$.
  6. $\det(AB) = \det(A)\det(B)$.
  7. $\det(A^{-1}) = \frac{1}{\det(A)}$.
- Adjoint: Transpose of cofactor matrix.
- $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$
- Cramer's Rule: $x = \frac{1}{\det(A)} \begin{bmatrix} \det(A_1) \\ \det(A_2) \\ \ldots \end{bmatrix}$, given A is invertible.

Vector Spaces

- Lines in $\mathbb{R}^2$:
  1. Implicit: $\{(x,y) | ax + by = c\}$.
  2. Explicit: $\{(t, \frac{c-at}{b}) | t \in \mathbb{R}\}$.
- Planes in $\mathbb{R}^3$:
  1. Implicit: $\{(x,y,z) | ax + by + cz = d\}$.
  2. Explicit: $\{(s, t, \frac{d-as-bt}{c}) | s, t \in \mathbb{R}\}$.
- Lines in $\mathbb{R}^3$:
  1. Implicit: $\{(x,y,z) | a_1x + b_1y + c_1z = d_1 \,\&\, a_2x + b_2y + c_2z = d_2\}$.
  2. Explicit: $\{(a_0 + at, b_0 + bt, c_0 + ct | t \in \mathbb{R}\}$.
- Linear Combination: For $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k} \in \mathbb{R}^n$, for any $c_1, c_2, \ldots, c_n \in \mathbb{R}$, the vector $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$ is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$.
- Span(S): The set of all linear combinations of $u_1, u_2, \ldots, u_k \in S$.

- Subspace: V is a subspace of $\mathbb{R}^n$ if $V = Span(S)$.
  1. $\mathbf{0} \in V$.
  2. For any vector $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k} \in V$ and any real number $c_1, c_2, \dots, c_k \in \mathbb{R}$, $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} \in V$.
- Linear Independence: $\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_k}$ are linearly independent if the equation $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \mathbf{0}$ only has the trivial solution (no redundant vector).
- Basis: S is a basis for V is S spans V and S is linearly independent.
- The empty set is the basis for zero space.
- Dimension: Number of vectors in the basis.
- Invertibility:
  1. $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.
  2. $rref(A) = I$.
  3. A can be expressed as a product of elementary matrix.
  4. $\det(A) \neq 0$.
  5. The rows of A forms a basis of $\mathbb{R}^n$.
  6. The columns of A forms a basis of $\mathbb{R}^n$.
- Transition Matrix: $\{\boldsymbol{v_1}\ \boldsymbol{v_2}\ \boldsymbol{v_3} | \boldsymbol{u_1} | \boldsymbol{u_2} | \boldsymbol{u_3}\}$.

Vector Spaces Associated with Matrices

- Row Space: $Span\{\boldsymbol{r_1}, \boldsymbol{r_2}, \dots, \boldsymbol{r_n}\}$.
- Column Space: $Span\{\boldsymbol{c_1}, \boldsymbol{c_2}, \dots, \boldsymbol{c_n}\}$.
- Pivot columns form a basis of column space.
- $rank(A) = \dim(R) = \dim(C)$.
- $rank(AB) \leq \min\{rank(A), rank(B)\}$.
- $rank(A + B) \leq rank(A) + rank(B)$.
- Dimension Theorem: $rank(A) + nullity(A) =$ number of columns of A.

Orthogonality

- $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if and only if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.
- If S is an orthogonal set of non-zero vectors, then S is linearly independent.
- Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_n}\}$ be an orthogonal basis for V, then for any vector $\boldsymbol{w} \in V$, $\boldsymbol{w} = \frac{\boldsymbol{w}\cdot\boldsymbol{u_1}}{\boldsymbol{u_1}\cdot\boldsymbol{u_1}}\boldsymbol{u_1} + \frac{\boldsymbol{w}\cdot\boldsymbol{u_2}}{\boldsymbol{u_2}\cdot\boldsymbol{u_2}}\boldsymbol{u_2} + \cdots + \frac{\boldsymbol{w}\cdot\boldsymbol{u_n}}{\boldsymbol{u_n}\cdot\boldsymbol{u_n}}\boldsymbol{u_n}$.
- Orthogonality: $\boldsymbol{u}$ is orthogonal to V if it is orthogonal to every vector in V.
- $proj_v(\boldsymbol{w}) = \frac{\boldsymbol{w}\cdot\boldsymbol{u_1}}{\boldsymbol{u_1}\cdot\boldsymbol{u_1}}\boldsymbol{u_1} + \frac{\boldsymbol{w}\cdot\boldsymbol{u_2}}{\boldsymbol{u_2}\cdot\boldsymbol{u_2}}\boldsymbol{u_2} + \cdots + \frac{\boldsymbol{w}\cdot\boldsymbol{u_n}}{\boldsymbol{u_n}\cdot\boldsymbol{u_n}}\boldsymbol{u_n}$.
- Gram-Schmidt Process:
  Let $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_n}\}$ be a basis of V, then:
  $\boldsymbol{v_1} = \boldsymbol{u_1}$,
  $\boldsymbol{v_2} = \boldsymbol{u_2} - \frac{\boldsymbol{u_2}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1}$,
  …,
  $\boldsymbol{v_n} = \boldsymbol{u_n} - \frac{\boldsymbol{u_n}\cdot\boldsymbol{v_1}}{\boldsymbol{v_1}\cdot\boldsymbol{v_1}}\boldsymbol{v_1} - \cdots$.
  $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}\}$ forms an orthogonal basis of V (can be further normalised).
- Least Square Solution: $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$.
- Orthogonal Matrix: $A^{-1} = A^T$.

Diagonalisation

- Eigenvectors and Eigenvalues: $A\boldsymbol{u} = \lambda\boldsymbol{u}$.

- Characteristic Equation: $\det(\lambda I - A) = 0$.
- For triangular matrix A, $\det(\lambda I - A) = (\lambda - a_{11})(\lambda - a_{22}) \dots (\lambda - a_{nn})$.
- Diagonalisable: $\exists P$ such that $P^{-1}AP$ is diagonal.
- A is diagonalisable if and only if A has n linearly independent eigenvectors.
- If A has n distinct eigenvalues then A is diagonalisable.
- Orthogonally Diagonalisable: $\exists$orthogonal matrix P such that $P^{-1}AP$ is diagonal.
- A is orthogonally diagonalisable if and only if A is symmetric.

Linear Transformation

- Linear Transformation: $T(c\boldsymbol{u} + d\boldsymbol{v}) = cT(\boldsymbol{u}) + dT(\boldsymbol{v})$. $T(\boldsymbol{0}) = \boldsymbol{0}$.
- Scaling: $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.
- Reflection: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- Rotation: $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$.
- Translation is not a linear transformation.
- Shears: $\begin{bmatrix} 1 & 0 & k_1 \\ 0 & 1 & k_2 \\ 0 & 0 & 1 \end{bmatrix}$.