MA1102R Calculus (Comprehensive)

Link: tinyurl.com/tx-ma1102r

For inquiries, please kindly contact the author via email tianxiaos1202@gmail.com.

Functions

- **Set Operations:** $A \cup B$, $A \cap B$, $A \setminus B$, $A \times B$
- **Common Sets:** Z, N, Q, R, $\varnothing$
- **Algebra of Functions:** Addition; Subtraction; Multiplication; Division; Composite
- **Type of Functions:** Absolute Value Functions; Polynomials; Rational Functions ($f(x) = \frac{P(x)}{Q(x)}$); Trigonometric Functions; Power Functions (x^n)
- **Parity of Functions**
  - If $f(x) = -f(-x)$, then *f* is odd.
  - If $f(x) = f(-x)$, then *f* is even.
- **1-to-1 Function:** If $f(a) = f(b)$, then $a = b$.

Limits

- **Intuitive Definition of Limits**

  If by taking *x* to be sufficiently close to *a*, the value of *f(x)* is arbitrarily close to the number *L*, then the limit of *f(x)*, as *x* approaches *a*, is *L*.
  - $\lim_{x \to a} f(x) = L$ or $x \to a \Rightarrow f(x) \to L$
- **Methods of Finding Limits**
  - Direct Substitution
  - If *f(x)* = *g(x)* for all values of *x* near *a*, $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$.
  - Squeeze Theorem

    If $f(x) \le g(x) \le h(x)$ for all values of x near a, and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$. (e.g. $-1 \le \sin x \le 1$)
- If $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x) = 0$.
- **One-sided Limit:** $\lim_{x \to a+} f(x)$ is the limit of *f(x)* as *x* approaches *a* from the right, vice versa.
- **Infinite Limit**

  - Definition: As $x \to a$, $f(x)$ is arbitrarily large/small.
  - Infinite limit does not exist.

- **Precise Definition of Limit**

  $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| > \varepsilon$.

  Presentation

  For $\varepsilon > 0$, choose $\delta = g(\varepsilon)$. Then whenever $0 < |x - a| < \delta$, $|f(x) - L| > \varepsilon$.
  (Always try to express $|f(x) - L|$ in terms of $|x - a|$.)

- **Precise Definition of Infinite Limit**

  $\lim_{x\to a} f(x) = \infty$ if for every $\varepsilon > 0$, there exists a number $M > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) > M$.

- **Precise Definition of Limit at Infinity**

  $\lim_{x\to \infty} f(x) = L$ if for every $\varepsilon > 0$, there exists a number $N > 0$ such that $x > N \Rightarrow |f(x) - L| > \varepsilon$.

- **Triangle Inequality:** $|x| - |y| \le |x + y| \le |x| + |y|$

Continuity

- A function *f* is said to be continuous at *a* if:
  - *f(a)* is well-defined;
  - $\lim_{x\to a} f(x)$ exists;
  - $\lim_{x\to a} f(x) = f(a)$
- **Type of Discontinuities:** Infinite Discontinuity; Jump Discontinuity
- **Intermediate Value Theorem**

  Let *f* be a function continuous on [*a, b*]. Suppose that $f(a) \neq f(b)$ and *N* is between *f(a)* and *f(b)*, then there exists $c \in (a, b)$ such that *f(c)* = *N*.

Derivatives

- **Definition:** $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$
- **Existence:** The derivative of *f(x)* at *a* exists if the limit *exists*.
- **Differentiation Formulas**

- $(f+g)' = f' + g'$
- $(fg)' = f'g + fg'$
- $(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$
- $(f \circ g)' = f'(g) \times g'$ (Chain Rule)
- $\frac{d}{dx}x^n = nx^{n-1}$
- $\frac{d}{dx}sin\ x = cos\ x$; $\frac{d}{dx}cos\ x = sin\ x$; $\frac{d}{dx}tan\ x = sec^2x$; $\frac{d}{dx}cot\ x = -\ csc^2x$; $\frac{d}{dx}sec\ x = sec\ x\ tan\ x$; $\frac{d}{dx}csc\ x = -\ csc\ x\ cot\ x$

- **Implicit Differentiation**
  a. Differentiate *f(x, y) = 0* with respect to *x*, regarding *y* as a differentiable function in *x*.
  b. Solve $\frac{dy}{dx}$ in terms of *x* and *y*.
- **Higher Order Derivatives**
  - Zeroth Derivative: $f^{(0)} = f$
  - *n*-th Derivative: $f^{(n)}(x) = (f^{(n-1)})'(x) = \frac{d^n y}{dx^n}$ (Suppose *f* is *n* times differentiable.)

Application of Derivatives

- **Extreme Values**

  Let *f* be a function with domain *D*.

  f is said to have an absolute maximum value at $c \in D$ if *f(c) > f(x)* for every $x \in D$.

  f is said to have an absolute minimum value at $c \in D$ if *f(c) < f(x)* for every $x \in D$.

  - Fermat's Theorem: Let *f* be a function such that *f* has a local extreme value at *c* and *f* is differentiable at *c*, then *f'(c) = 0.*
    - If *f'(c) = 0* or does not exist, *c* is called a critical point.
    - If *f'(c) = 0*, *c* is called a stationary point.

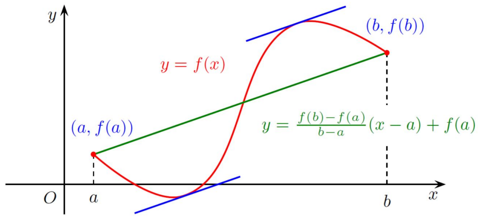
Presentation - Closed Interval Method

Let *f* be continuous on [*a, b*].

1. Evaluate *f(a)* and *f(b)*.
2. Find the critical points on (*a, b*).
3. Evaluate the value of *f* at the critical points.
4. By comparing these values, find the absolute maximum and minimum values and the corresponding *x* value.

- **Mean Value Theorem**

  Suppose f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.



  - Rolle's Theorem

    Suppose f is continuous on [a, b] and differentiable on (a, b), and f(a) = f(b), then there exists $c \in (a, b)$ such that $f'(c) = 0$.

- **Increasing/Decreasing Test**

  Suppose f is continuous on $[a, b]$ and differentiable on (a, b):
  - If $f'(x) > 0$ for every x in (a, b), then f is increasing on $[a, b]$.
  - If $f'(x) < 0$ for every x in (a, b), then f is decreasing on $[a, b]$.

- **Local Extreme Value**
  - First Derivative Test: Find for each critical point the sign of f' before and after the point.

    Example

    The critical point of $f(x) = x^2 - 2x + 1$ is $x = 1$.

Intervals	$(-\infty, 1)$	$(1, \infty)$
$f'(x)$	< 0	> 0
$f(x)$	↘	↗

    Hence we conclude that $x = 1$ is a local minimum.

  - Second Derivative Test: Find for each critical point the sign of f''.

$f''(x) > 0$	Local Minimum
$f''(x) < 0$	Local Maximum
$f''(x) = 0$	Inflection Point

    - Second derivative test is inconclusive if $f'(x) = f''(x) = 0$.

- **Concavity**

- $f''(x) > 0 \Rightarrow f$ is concave upwards.
- $f''(x) < 0 \Rightarrow f$ is concave downwards.

- **l'Hôpital's Rule (0/0 or ∞/∞)**

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- l'Hôpital's Rule is inconclusive if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ does not exist and is not $\pm\infty$.

Integrals

- **Riemann Sum**

Presentation - Riemann Sum

To find $\int_a^b f(x)$ using Riemann Sum:

1. Divide $[a, b]$ into n equal subintervals: $[x_0, x_1]$, $[x_1, x_2]$, …, $[x_{n-1}, x_n]$.
2. The length of each subinterval, $\Delta x = \frac{b-a}{n}$.
3. From each subinterval, choose a sample point x_i^*.
4. The Riemann Sum is given by $S_n = [f(x_1^*) + f(x_2^*) + ... + f(x_n^*)]\Delta x$.
5. Therefore, $\int_a^b f(x) = \lim_{n \to \infty} S_n$.

- Inverse Use of Riemann Sum

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(\tfrac{i}{n})(\tfrac{1}{n}) = \int_0^1 f(x)dx$$

- **Properties of Definite Integral**
  - $\int_a^b f(x) = -\int_b^a f(x)$
  - Let f be a continuous function on $[a, b]$, $c \in (a, b)$, then $\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$

- **Fundamental Theorem of Calculus**
  - Let f be a continuous function on $[a, b]$. Let $g(x) = \int_a^x f(t)dt$. Then:
    - g is continuous on $[a, b]$;
    - g is differentiable on $[a, b]$;
    - $g'(x) = f(x)$ for every $x \in (a, b)$;
    - $\frac{d}{dx}\int_a^x f(t)dt = f(x)$
      -
  - Mean Value Theorem

    Let f be a continuous function on $[a, b]$. There exists $c \in (a, b)$ such that

$\int_a^b f(x)dx = (b-a)f(c)$.

- Let f be a continuous function on $[a, b]$. Let $F'(x) = f(x)$ for all $x \in (a, b)$. Then $\int_a^b f(x)dx = F(b) - F(a)$. F is also called an indefinite integral of f.

- **Improper Integral**
  - Suppose f is continuous at $[a, b)$ but not b, $\int_a^b f(x)dx = \lim_{x \to b-} \int_a^x f(x)dx$.
  - Suppose f is continuous at $(a, b]$ but not a, $\int_a^b f(x)dx = \lim_{x \to a+} \int_x^b f(x)dx$.
  - Suppose f is continuous at (a, b) but not $c \in (a, b)$, $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$. It is convergent only if both limits exist.
  - The same applies to infinity.

Inverse Functions and Transcendental Functions

- **Inverse Functions**
  - Let f be a one-to-one function. Then $f(x) = y \Leftrightarrow f^{-1}(y) = x$.
  - Graphs of f and f^{-1} is symmetric about the line $y = x$.
  - Given $f(a) = b$, $f^{-1\prime}(b) = \frac{1}{f'(a)}$.
  - Inverse trigonometric function:

Function	Derivative	Domain
$sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$	$(-1, 1)$
$cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$	$(-1, 1)$
$tan^{-1}(x)$	$\frac{1}{1+x^2}$	$(-\infty, +\infty)$
$cot^{-1}(x)$	$-\frac{1}{1+x^2}$	$(-\infty, +\infty)$
$sec^{-1}(x)$	$\frac{1}{x\sqrt{x^2-1}}$	$(-\infty, -1) \cup (1, +\infty)$
$csc^{-1}(x)$	$-\frac{1}{x\sqrt{x^2-1}}$	$(-\infty, -1) \cup (1, +\infty)$

  - $tan^{-1}(x) + cot^{-1}(x) = \frac{\pi}{2}$
  - $sec^{-1}(x) + csc^{-1}(x) = \frac{\pi}{2}$ if $x \geq 1$ else $\frac{5\pi}{2}$ if $x \leq -1$
- **Logarithmic and Exponential Functions**
  - Definition of ln: $ln(x) = \int_1^x \frac{1}{t}dt$
  - $\frac{d}{dx}ln(x) = \frac{1}{x}$; $\frac{d}{dx}exp(x) = exp(x)$
  - $ln(x)$ and $exp(x)$ are inverse functions to each other.
  - $lim\ exp(f(x)) = exp(lim(f(x)))$
- **Hyperbolic Trigonometric Functions**
  - Exponential form and derivative of hyperbolic trigonometric functions:

Function	Exponential Form	Derivative
$sinh(x)$	$\frac{e^x-e^{-x}}{2}$	$cosh(x)$
$sinh^{-1}(x)$	N.A.	$\frac{1}{\sqrt{1+x^2}}$
$cosh(x)$	$\frac{e^x+e^{-x}}{2}$	$sinh(x)$ for $x > 0$
$cosh^{-1}(x)$	N.A.	$\frac{1}{\sqrt{x^2-1}}$ for $x > 1$

Integration Techniques (See Appendix)

- **Substitution Rule**

  Let $u = g(x)$, then $\int f(g(x))dx = \int f(u)\frac{dx}{du}du$.

- **Inverse Substitution Rule**

  Let $x = g(u)$, then $\int f(x)dx = \int f(g(u))\frac{dx}{du}du$.

  - Trigonometric substitution
    1. $\sqrt{a^2 - x^2}$, substitute $x = asin\ t$.
    2. $\sqrt{a^2 + x^2}$, substitute $x = atan\ t$.
    3. $\sqrt{x^2 - a^2}$, substitute $x = asec\ t$.
  - Universal trigonometric substitution

    Let $t = tan(\frac{x}{2})$, then $\frac{dx}{dt} = \frac{2}{1+t^2}$. Then $(sin\ x,\ cos\ x) = (\frac{2t}{1+t^2},\ \frac{1-t^2}{1+t^2})$.

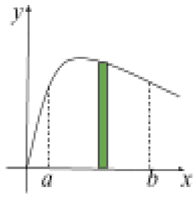
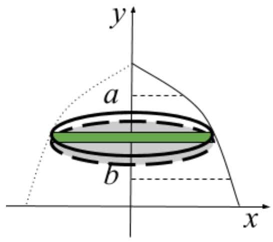
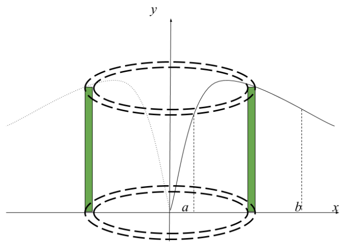
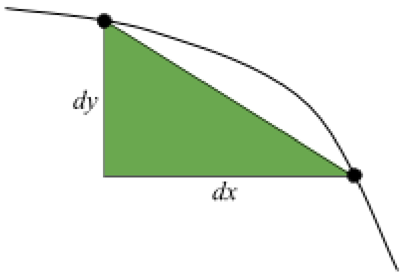
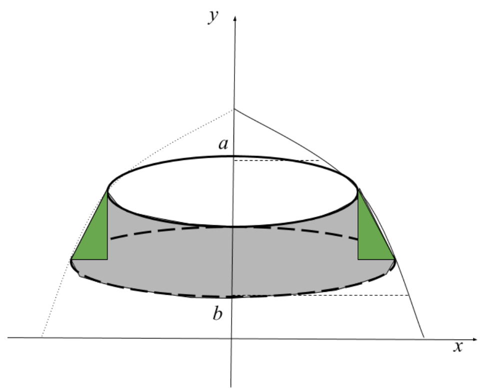
- **Integration by Parts**

  $\int u\ \frac{dv}{dx}dx = uv - \int v\ \frac{du}{dx}dx$

  - Recursive formula of trigonometric integration
    1. $\int sin^n x\ dx = -\frac{1}{n}sin^{n-1}xcos\ x + \frac{n-1}{n}\int sin^{n-2}x\ dx$
    2. $\int cos^n x\ dx = \frac{1}{n}cos^{n-1}xsin\ x + \frac{n-1}{n}\int cos^{n-2}x\ dx$
    3. $\int tan^n x\ dx = \frac{1}{n-1}tan^{n-1}x - \int tan^{n-2}x\ dx$
    4. $\int cot^n x\ dx = -\frac{1}{n-1}cot^{n-1}x - \int cot^{n-2}x\ dx$
    5. $\int sec^n x\ dx = \frac{1}{n-1}sec^{n-2}xtan\ x + \frac{n-2}{n-1}\int sec^{n-2}x\ dx$
    6. $\int csc^n x\ dx = -\frac{1}{n-1}csc^{n-2}xcot\ x + \frac{n-2}{n-1}\int csc^{n-2}x\ dx$

- **Partial Fractions**

Application of Definite Integrals

• **Area**	 $A = \int_a^b y\, dx$	
• **Volume**	 $V = \int_b^a \pi x^2\, dy$	 $V = \int_a^b 2\pi xy\, dx$
• **Arc Length**	 $L = \sqrt{dx^2 + dy^2}$	
• **Surface Area**	 $A = \int_b^a 2\pi xL$	

First Order Ordinary Differential Equations

- **Common Techniques**
  - $\frac{dy}{dx} = f(x) \Rightarrow y = \int f(x)\, dx$
  - $\frac{dy}{dx} = g(y) \Rightarrow x = \int \frac{1}{g(y)}\, dy$

- $\frac{dy}{dx} = f(x)\,g(y) \Rightarrow \int f(x)\,dx = \int \frac{1}{g(y)}\,dy$
- $\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow Integrating\,factor\ v(x) = e^{P(x)} \Rightarrow y = \frac{1}{v(x)} \int v(x)Q(x)\,dx$
- $\frac{dy}{dx} + P(x)y = Q(x)y^n \Rightarrow z = y^{1-n},\ \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx} \Rightarrow \frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$

Good luck!

Appendix

Common Differentiation and Integration Results

Differentiation	**Integration (w/o + *C*)**
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n\ dx = \frac{1}{n+1}x^{n+1}$
$\frac{d}{dx}(ln\ x) = \frac{1}{x}$	$\int \frac{1}{x}\ dx = ln\ x$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x$
$\frac{d}{dx}(a^x) = ln\ a \times a^x$	$\int a^x dx = \frac{1}{ln\ a}a^x$
$\frac{d}{dx}((sin\ x,\ cos\ x)) = (cos\ x,\ -sin\ x)$	$\int(sin\ x,\ cos\ x)\ dx = (-cos\ x,\ sin\ x)$
$\frac{d}{dx}((tan\ x,\ cot\ x)) = (sec^2x,\ -csc^2x)$	$\int(sec^2x,\ csc^2x)\ dx = (tan\ x,\ -cot\ x)$
$\frac{d}{dx}((sec\ x,\ csc\ x)) = (sec\ xtan\ x,\ -csc\ xcot\ x)$	$\int(sec\ xtan\ x,\ csc\ xcot\ x)\ dx = (sec\ x,\ -csc\ x)$
$\frac{d}{dx}((ln\ sin\ x,\ ln\ cos\ x)) = (cot\ x,\ -tan\ x)$	$\int(tan\ x,\ cot\ x)\ dx = (-ln\ \lvert cos\ x\rvert,\ ln\ \lvert sin\ x\rvert)$
$\frac{d}{dx}(ln\ (sec\ x + tan\ x)) = sec\ x$ $\frac{d}{dx}(ln\ (csc\ x - cot\ x)) = csc\ x$	$\int sec\ x\ dx = ln\ \lvert sec\ x + tan\ x\rvert$ $\int csc\ x\ dx = ln\ \lvert csc\ x - cot\ x\rvert$
$\frac{d}{dx}((sin^{-1}\frac{x}{a},\ cos^{-1}\frac{x}{a})) = (\frac{1}{\sqrt{a^2-x^2}},\ -\frac{1}{\sqrt{a^2-x^2}})$	$\int(\frac{1}{\sqrt{a^2-x^2}},\ \frac{1}{\sqrt{a^2-x^2}})\ dx = (sin^{-1}\frac{x}{a},\ -cos^{-1}\frac{x}{a})$
$\frac{d}{dx}((tan^{-1}\frac{x}{a},\ cot^{-1}\frac{x}{a})) = (\frac{a}{a^2+x^2},\ -\frac{a}{a^2+x^2})$	$\int(\frac{a}{a^2+x^2},\ \frac{a}{a^2+x^2})\ dx = (tan^{-1}\frac{x}{a},\ -cot^{-1}\frac{x}{a})$
$\frac{d}{dx}((sec^{-1}\frac{x}{a},\ csc^{-1}\frac{x}{a})) = (\frac{a}{x\sqrt{x^2-a^2}},\ -\frac{a}{x\sqrt{x^2-a^2}})$	$\int(\frac{a}{x\sqrt{x^2-a^2}},\ \frac{a}{x\sqrt{x^2-a^2}})\ dx = (sec^{-1}\frac{x}{a},\ -csc^{-1}\frac{x}{a})$