MA2101 Linear Algebra II

AY2021/22 Semester 1

Chapter 8 – General Vector Spaces

8.1. Fields

- (*Proposition 8.1.11*) Properties of the trace function:
  (a) $tr(A) + tr(B) = tr(A + B)$.
  (b) $tr(cA) = ctr(A)$.
  (c) $tr(AB) = tr(BA)$.

8.3. Subspaces

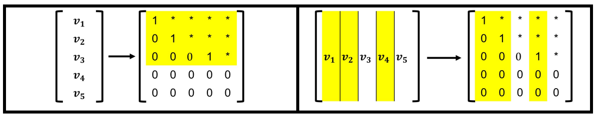
- Check whether W is a subspace of V:
  (a) $\mathbf{0} \in W$. Hence W is non-empty; AND
  (b) $\forall a, b \in \mathbb{F}, \boldsymbol{u}, \boldsymbol{v} \in W, a\boldsymbol{u} + b\boldsymbol{v} \in W$.

8.4. Linear Spans and Linear Independence

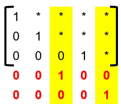
- Prove $\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_n}$ are linearly independent:
  The function $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_n\boldsymbol{u_n} = \mathbf{0}$ has only the trivial solution.

8.5. Bases and Dimensions

- $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.
- (*Theorem 8.5.15*) If W is a subspace of V, then:
  (a) $\dim(W) \leq \dim(V)$.
  (b) If $\dim(W) = \dim(V)$, then $W = V$.
- Find bases:



- Extend a set to a basis:



8.6. Direct Sums

- $W_1 + W_2$ is a direct sum if $W_1 \cap W_2 = \{\mathbf{0}\}$.

8.7. Cosets and Quotient Spaces

- Basis for V/W:
  Assume $V = span\{\boldsymbol{v_1}, \dots, \boldsymbol{v_k}, \boldsymbol{w_1}, \dots, \boldsymbol{w_m}\}$.
  Then $\{W + \boldsymbol{v_1}, \dots, W + \boldsymbol{v_n}\}$ forms a basis for V/W.

Chapter 9 – Linear Transformation

9.1. Linear Transformations

- Check whether T is a linear operator:
  For all $\boldsymbol{u}, \boldsymbol{v} \in V$ and $a, b \in \mathbb{F}$, check whether
  $$T(a\boldsymbol{u} + b\boldsymbol{v}) = aT(\boldsymbol{u}) + bT(\boldsymbol{v}).$$

9.2. Matrices for Linear Transformations

- Matrix for T relative to B and C:

$$[T]_{C,B} = \left[\begin{array}{c|c|c|c|c} [T(\boldsymbol{v_1})]_C & [T(\boldsymbol{v_1})]_C & [T(\boldsymbol{v_1})]_C & \cdots & [T(\boldsymbol{v_n})]_C \end{array}\right]$$

- Transition matrices from B to C:
  $$[\boldsymbol{u}]_C = [I_V]_{C,B}[\boldsymbol{u}]_B.$$

9.3. Compositions of Linear Transformations

- $[T \circ S]_{C,A} = [T]_{C,B}[S]_{C,A}$.
- $[T]_B = [I_V]_{B,C}[T]_C[I_V]_{C,B}$.

9.4. The Vector Spaces $\mathcal{L}(V, W)$

- Dimension of the set for all linear transformations from V to W:
  $$\dim(\mathcal{L}(V, W)) = \dim(V)\dim(W).$$

- Dual space: $\mathcal{L}(V, \mathbb{F}) = V^*$.
- $\dim(V) = \dim(V^*)$.

9.5. Kernels and Ranges

- (*Dimension Theorem for Matrices*) $rank(A) + nullity(A) = n$.
- (*Dimension Theorem for Linear Transformation*)
  $$\dim(V) = \dim(Ker(T)) + \dim(R(T)).$$

9.6. Isomorphisms

- Check whether T is an isomorphism:
  (a) T is a bijective mapping; OR
  (b) $[T]$ is invertible.
- Check whether V and W are isomorphic vector spaces:
  (a) There exists an isomorphism $T: V \to W$; OR
  (b) $\dim(V) = \dim(W)$. (*Theorem 9.6.13*)
- (*First Isomorphism Theorem*) $V/Ker(T) \cong R(T)$.