Compiled by Tian Xiao

MA2104 Multivariable Calculus

AY2020/21 Semester 2

1. Euclidean Spaces and Vector

- **Euclidean Spaces**
  - Basic Operators
    - Addition
    - Negation
    - Scalar Multiplication
    - Dot Product
      - Angle: $\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta$
      - Length: $|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$
      - Direction of $\boldsymbol{u}$: $\frac{\boldsymbol{u}}{|\boldsymbol{u}|}$
    - Cross Product
      - $\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
      - $\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$
      - $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$
      - Angle: $|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}|\sin\theta$.
      - Triple Product Formula: $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$
  - Line
    - Vector Equation: $\boldsymbol{r}(t) = \boldsymbol{r_0} + t\boldsymbol{v}$
    - Parametric Equation: $\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}$
  - Plane
    - Vector Equation: $(\boldsymbol{r} - \boldsymbol{r_0}) \cdot \boldsymbol{n} = 0$
    - Cartesian Equation: $Ax + By + Cz = D$
  - Sphere
    - Vector Equation: $|\boldsymbol{r} - \boldsymbol{r_0}| = a$
    - Cartesian Equation: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$

Triangle Inequality: $|\boldsymbol{u} + \boldsymbol{v}| \leq |\boldsymbol{u}| + |\boldsymbol{v}|$

Cauchy-Schwarz Inequality: $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq |\boldsymbol{u}||\boldsymbol{v}|$

2. Vector-Valued Single Variable Function

Parametrised Curve

- $\boldsymbol{r}: I \to \mathbb{R}^n$ parametrised by t: $\boldsymbol{r}(t) = \begin{pmatrix} r_0(t) \\ r_1(t) \\ r_2(t) \end{pmatrix}$.

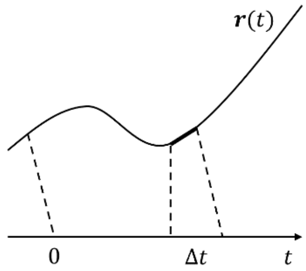
Limit

- $\boldsymbol{r}$ has limit $\boldsymbol{L}$ as $t \to t_0$ if and only if $\forall \epsilon \in \mathbb{R}_{\geq 0}$, $\exists \delta \in \mathbb{R}_{\geq 0}$, such that $\forall t \in I$ with $0 < |t - t_0| < \delta$, $|\boldsymbol{r}(t) - \boldsymbol{L}| < \epsilon$.
- $\lim_{t \to t_0} \boldsymbol{r}(t) = \boldsymbol{L} \Leftrightarrow \forall j \in \{1,2,\dots,n\}$, $\lim_{t \to t_0} r_j(t) = L_t$.

Continuity

- $\boldsymbol{r}(t)$ is continuous **at a point** $t = t_0$ if $\lim_{t \to t_0} \boldsymbol{r}(t) = \boldsymbol{r}(t_0)$.
- $\boldsymbol{r}(t)$ is continuous if it is continuous **at every point** in its domain.
- $\boldsymbol{r}(t)$ is continuous at $t = t_0$ if and only if every component function is continuous there.

Derivative



- $\boldsymbol{r}$ is differentiable **at a point** t if and only if $\exists \boldsymbol{r}'(t) \in \mathbb{R}^n$ such that $\lim_{\Delta t \to 0} \frac{1}{\Delta t}\big(\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)\big) = \boldsymbol{r}'(t)$ in $\mathbb{R}^n$.
- $\boldsymbol{r}$ is differentiable on I if $\boldsymbol{r}$ is differentiable **at every point** in its domain.
- $\boldsymbol{r}$ is differentiable at t if and only if every component function is differentiable there.
- $\boldsymbol{r}$ is continuously differentiable if and only if $\boldsymbol{r}$ is differentiable and $\boldsymbol{r}'$ is continuous.
- Sum Rule, Difference Rule, Scalar Multiplication Rule and Chain Rule apply.
- **Dot Product Rule**: $\frac{d}{dt}[\boldsymbol{u}(t) \cdot \boldsymbol{v}(t)] = \boldsymbol{u}'(t) \cdot \boldsymbol{v}(t) + \boldsymbol{u}(t) \cdot \boldsymbol{v}'(t)$.
- **Cross Product Rule**: $\frac{d}{dt}[\boldsymbol{u}(t) \times \boldsymbol{v}(t)] = \boldsymbol{u}'(t) \times \boldsymbol{v}(t) + \boldsymbol{u}(t) \times \boldsymbol{v}'(t)$

Integral

- $\boldsymbol{r}$ is integrable over $I = [a, b]$ if and only if $\exists \boldsymbol{L} \in \mathbb{R}^n$ such that $\lim_{||P|| \to 0} \sum_{k=1}^{n} \boldsymbol{r}(c_k)\Delta t_k = \boldsymbol{L}$.
- $\boldsymbol{r}$ is integrable over I if and only if every component function is integrable there.
- **Indefinite Integral $\boldsymbol{R}$**: $\forall x \in [a, b], \boldsymbol{R}(x) = \int_a^x \boldsymbol{r}(t)dt$.

3. Curve, Surface and Region

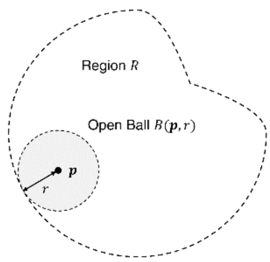
Curve

- **Smooth**: $\boldsymbol{r}$ is smooth if and only if $\boldsymbol{r}$ is continuously differentiable and has non-vanishing derivative.
  - Non-vanishing: $\forall t \in I, \boldsymbol{r}'(t) \neq \boldsymbol{0}$.
- **Piecewise Smooth**: Finite number of smooth curves pieced together in a continuous fashion.
- **Arc Length**: $\int_a^b |\boldsymbol{r}'(t)|dt$.

Surface

- **Parametrised Surface**: $\boldsymbol{r}: R \to \mathbb{R}^n$, where $R \in \mathbb{R}^2$ is an open rectangle, a closed rectangle or a region.
- **Smooth**: $\boldsymbol{r}$ is smooth if and only if $\boldsymbol{r}$ is continuously differentiable and has non-vanishing $\boldsymbol{r_u} \times \boldsymbol{r_v}$.
- **Area**: $\iint_R |\boldsymbol{r_u} \times \boldsymbol{r_v}|dA$.

Region



- A region in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$, which is usually assumed to be "nice" – connected, open/compact, etc.
  - A region R is **open** if and only if $\forall \boldsymbol{p} \in R$, $\exists r \in \mathbb{R}_{\geq 0}$ such that $B(\boldsymbol{p}, r) \subseteq R$.
  - A region R is **bounded** if it lies inside a disk of finite radius.
  - A region R is **compact** if it is closed and bounded.
  - A **compact rectangle** is a subset of $\mathbb{R}^n$ in the form $X = X_1 \times X_2 \times \ldots \times X_n$, where each X_i is a closed and bounded interval in $\mathbb{R}^1$ (i.e. $[a, b]$).

4. Multivariable Function

Multivariable Function

- $f: R \to \mathbb{R}^n$ where $R \in \mathbb{R}^m$ is an open region.
- Vector of Scalar-Valued Component Functions: $f(\boldsymbol{p}) = \begin{pmatrix} f_1(\boldsymbol{p}) \\ \dots \\ f_n(\boldsymbol{p}) \end{pmatrix}$.

Limit

- $\boldsymbol{f}$ has limit $\boldsymbol{L}$ as $\boldsymbol{p} \to \boldsymbol{p_0}$ if and only if $\forall \epsilon \in \mathbb{R}_{\geq 0}$, $\exists \delta \in \mathbb{R}_{\geq 0}$, such that $\forall t \in I$ with $0 < |\boldsymbol{p} - \boldsymbol{p_0}| < \delta$, $|f(\boldsymbol{p}) - \boldsymbol{L}| < \epsilon$.
- **Calculation Rules** (for all functions): Let $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} f(\boldsymbol{p}) = \boldsymbol{L}$, $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} g(\boldsymbol{p}) = \boldsymbol{M}$, then:
  - $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} (f(\boldsymbol{p}) \pm g(\boldsymbol{p})) = \boldsymbol{L} \pm \boldsymbol{M}$;
  - $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} cf(\boldsymbol{p}) = c\boldsymbol{L}$.
- **Calculation Rules** (for all scalar-valued functions): Let $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} f(\boldsymbol{p}) = L$, $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} g(\boldsymbol{p}) = M$, then:
  - $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} (f(\boldsymbol{p}) \cdot g(\boldsymbol{p})) = L \cdot M$;
  - $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} (\frac{f(\boldsymbol{p})}{g(\boldsymbol{p})}) = \frac{L}{M}$, where $M \neq 0$;
  - $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} \left(f(\boldsymbol{p})\right)^n = L^n$, where n is a positive integer;
  - $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} (\sqrt[n]{f(\boldsymbol{p})}) = \sqrt[n]{L}$, where n is a positive integer.

Continuity

- $f(\boldsymbol{p})$ is continuous **at a point** $\boldsymbol{p} = \boldsymbol{p_0}$ if $\lim_{\boldsymbol{p} \to \boldsymbol{p_0}} f(\boldsymbol{p}) = f(\boldsymbol{p_0})$.
- $f(\boldsymbol{p})$ is continuous on R if it is continuous **at every point** on R.
- If f is continuous at $\boldsymbol{p_0}$, g is continuous at $f(\boldsymbol{p_0})$, then $g \circ f$ is continuous at $\boldsymbol{p_0}$.

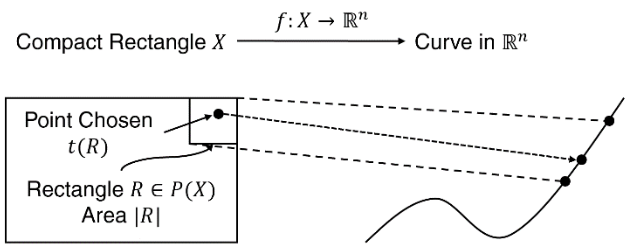
Differentiability

- f is differentiable **at a point** $\boldsymbol{p_0}$ if and only if $\exists A: \mathbb{R}^m \to \mathbb{R}^n$ such that $\forall \epsilon \in \mathbb{R}_{\geq 0}$, $\exists \delta \in \mathbb{R}_{\geq 0}$, such that $\forall \boldsymbol{p} \in R$ with $|\boldsymbol{p} - \boldsymbol{p_0}| < \boldsymbol{\delta}$, one has $\left|f(\boldsymbol{p}) - \left(f(\boldsymbol{p_0}) + A(\boldsymbol{p} - \boldsymbol{p_0})\right)\right| \leq \epsilon|\boldsymbol{p} - \boldsymbol{p_0}|$. Here
- f is differentiable on R if it is differentiable **at every point** on R.
- Differentiability implies continuity.
- **Directional Derivative w.r.t. $\boldsymbol{u}$**: $(D_{\boldsymbol{u}} f)(\boldsymbol{p_0}) = \lim_{s \to 0} \frac{f(\boldsymbol{p_0} + s\boldsymbol{u}) - f(\boldsymbol{p_0})}{s} = A\boldsymbol{u}$.
- $(D_{\boldsymbol{u}} f)(\boldsymbol{p_0}) = \begin{pmatrix} (D_{\boldsymbol{u}} f_1)(\boldsymbol{p_0}) \\ \dots \\ (D_{\boldsymbol{u}} f_n)(\boldsymbol{p_0}) \end{pmatrix}$.
- **Partial Derivative**: Directional derivative w.r.t. standard unit vectors, $\frac{\partial f}{\partial x_j}(\boldsymbol{p_0}) = f_{x_j}(\boldsymbol{p_0}) = \lim_{s \to 0} \frac{f(\boldsymbol{p_0} + s e_j) - f(\boldsymbol{p_0})}{s}$.
- $(Df)(\boldsymbol{p_0}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{p_0}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{p_0}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\boldsymbol{p}_0) & \cdots & \frac{\partial f_n}{\partial x_n}(\boldsymbol{p_0}) \end{bmatrix} = J_f(\boldsymbol{p_0})$.

- **Gradient of Scalar-Valued Function**: $(\Delta f)(\boldsymbol{p}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\boldsymbol{p}) \\ \cdots \\ \frac{\partial f}{\partial x_n}(\boldsymbol{p}) \end{pmatrix}$. This is obviously same as first row of total derivative map. Hence, $(\Delta f)(\boldsymbol{p}) \cdot \boldsymbol{u} = (D_{\boldsymbol{u}} f)(\boldsymbol{p})$.
- Sum Rule, Difference Rule, Scalar Multiplication Rule and Chain Rule apply.
- f is continuously differentiable on R if and only if f is differentiable on R and f' is continuous.
- f is continuously differentiable on R if and only if all partial derivatives of every component function exist and is continuous.
- f is of class C^r if and only if all partial derivatives of every component function exist and is of class C^{r-1}.
- **Taylor's Theorem**: Let R be an open subset of $\mathbb{R}^m$, $f: R \to \mathbb{R}$ be a scalar-valued function of class C^{r+1}. Let $\boldsymbol{p_0} \in \boldsymbol{R}$ and suppose $\delta \in \mathbb{R}_{\geq 0}$ such that $B(\boldsymbol{p_0}, \delta) \subseteq R$ (in domain). Then, $\forall \boldsymbol{\xi} \in \mathbb{R}^m$ with $|\boldsymbol{\xi}| < \delta$, one has $f(\boldsymbol{p_0} + \boldsymbol{\xi}) = \left[\sum_{d=0}^{r} \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{Z}_{\geq 0}^m \\ |\boldsymbol{\alpha}| = d}} \frac{1}{\boldsymbol{\alpha}!} \frac{\partial^d f}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}(\boldsymbol{p_0}) \cdot \left(\xi_1^{\alpha_1} \ldots \xi_m^{\alpha_m}\right)\right] + R(\boldsymbol{\xi})$, where $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \cdots \\ \alpha_m \end{pmatrix}$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_m$, $\boldsymbol{\alpha}! = \alpha_1! \ldots \alpha_m!$. Here, $\exists c \in (0,1)$ (so that it is inside the open ball) such that $R(\boldsymbol{\xi}) = \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{Z}_{\geq 0}^m \\ |\boldsymbol{\alpha}| = r+1}} \frac{1}{\alpha!} \frac{\partial^d f}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}(\boldsymbol{p_0} + c\boldsymbol{\xi}) \cdot \left(\xi_1^{\alpha_1} \ldots \xi_m^{\alpha_m}\right)$.
  - Case $m = 1$: $\exists c \in (a, b)$ such that $f(b) = \left[f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n\right] + \left[\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}\right]$.
  - Case $m = 2$: $\exists c \in (0,1)$ such that $f(a+h, b+k) = \Big[f(a,b) + \left(hf_x + kf_y\right)|_{a,b} + \frac{1}{2!}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)|_{a,b} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f|_{a,b}\Big] + [\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f|_{a+ch, b+ck}]$.
  - Proof is simple by induction.
- **Linear Approximation**: $f(\boldsymbol{p}) \approx L(\boldsymbol{p}) = f(\boldsymbol{p_0}) + \sum_{k=1}^{m} f_{x_k}(\boldsymbol{p_0})(p_k - p_{0_k})$.
  - Error: Let $\boldsymbol{\xi} = \boldsymbol{p} - \boldsymbol{p_0}$, then $E(\boldsymbol{p}) \leq \frac{1}{2}M(\sum_{i=1}^{m} |\xi_i|^2)$, where $\forall i, j \in \{1, \ldots, m\}, \forall \boldsymbol{p} \in R, f_{x_i x_j}(\boldsymbol{p}) \leq M$.
  - Proof is via Taylor's Theorem (Case $n = 1$).

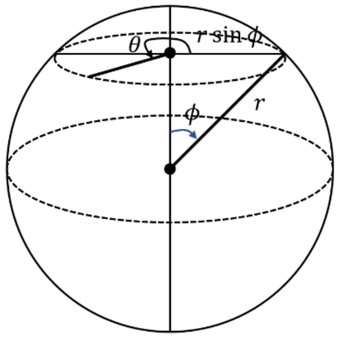
<u>Integral</u>

- f is Riemann integrable over X if and only if $\exists \boldsymbol{L} \in \mathbb{R}^n$ such that $\lim_{||P|| \to 0} \sum_{R \in P} f\left(t(R)\right)|R| = \boldsymbol{L}$.
- Sum Rule, Difference Rule, Scalar Multiplication Rule apply.
- **Domination Rule**: $\int$ is order-preserving.



- **Additivity Rule**: $\int_R f dA = \int_{R_1} f dA + \int_{R_2} f dA$ if $R = R_1 \cup R_2$ and $|R_1 \cap R_2| = 0$.
- f is Riemann integrable if and only if every component function is Riemann integrable.
- A function f is Riemann integrable if and only if f is bounded and $Dis(f)$ is of measure 0 in $\mathbb{R}^m$.
  - Measure 0: $\forall \epsilon \geq 0$, $\exists$rectangles $R_1, \ldots, R_n$ such that $Dis(f) \subseteq R_1 \cup \ldots \cup R_n$ and $|R_1| + \cdots + |R_n| < \epsilon$.
  - **Every continuous function on X is Riemann integrable.**
- $R \subseteq \mathbb{R}^m$ is a "nice" region if R us closed, bounded and the set of boundary points is of measure 0.

- **Fubini's Theorem**: Given R is a nice region and f is continuous on R, then $\int_{X\times Y} f(x,y)d(x,y) = \int_X \int_Y f(x,y)dy\,dx = \int_Y \int_X f(x,y)dx\,dy$.
- **Volume**: $vol(R) = \int_R 1d\boldsymbol{x}$.
- **Average Value**: $\frac{\int_R f dx}{\int_R 1 dx}$.
- **Change of Variable Formula**: $\int_R f(\boldsymbol{x})d\boldsymbol{x} = \int_G f\big(g(\boldsymbol{u})\big)|J_g(\boldsymbol{u})|d\boldsymbol{u}$.
- **Polar Integral**: $\int_R f(x,y)dxdy = \int_G f(r\cos\theta, r\sin\theta)\, r\, drd\theta$.
- **Cylindrical Integral**: $\int_R f(x,y,z)dxdydz = \int_G f(r\cos\theta, r\sin\theta, z)\, r\, drd\theta dz$.



- **Spherical Integral**: $\int_R f(x,y,z)dxdydz = \int_G f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho d\phi d\theta$.

5. Line Integral

Line Integral Of Scalar-Valued Function

- Let C be a smooth curve in $\mathbb{R}^n$, and $\boldsymbol{r}(t)$ is a bijective smooth parametrisation of C. Let $f: C \to \mathbb{R}$ be a scalar-valued function on C. Then $\int_C f\, ds = \int_I f\big(\boldsymbol{r}(t)\big)|\boldsymbol{r}'(t)|dt$.
- **Additivity Rule**: If a piecewise smooth curve C is made up of finite number of smooth curves $C_1, C_2, \dots, C_n$, then $\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \dots + \int_{C_n} f\, ds$.

Vector Field and Gradient

- A vector field on $X \subseteq \mathbb{R}^n$ is a vector-valued function $\boldsymbol{F}: X \to \mathbb{R}^n$.
- A vector field is continuous/smooth if and only if $\boldsymbol{F}$ is continuous/smooth.
- **Gradient**: $\nabla f(\boldsymbol{p}) = (\frac{\partial f}{\partial x_1}(\boldsymbol{p}), \frac{\partial f}{\partial x_2}(\boldsymbol{p}), \dots, \frac{\partial f}{\partial x_n}(\boldsymbol{p}))$. f is a potential function of ∇f.
- Δf is a continuous vector field if and only if f is continuously differentiable.
- Sum Rule, Difference Rule, Constant Multiple Rule, Product Rule and Quotient Rule apply.
- Conservative fields are gradient fields, $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$ is path independent on conservative fields.
- If $\boldsymbol{F}$ is conservative on D, then $\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = 0$ around every loop in D.
- **Component Test for Conservative Fields**: If $\boldsymbol{F}$ id s gradient vector field, then $\forall i,j \in \{1,2,\dots,n\}$, $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ on D. If D is connected and simply connected, then the converse holds.
  - In $\mathbb{R}^3$, let $\boldsymbol{F} = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$ be a field on an open simply connected domain, then $\boldsymbol{F}$ is conservative if and only if $\begin{cases} \frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \\ \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \\ \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \end{cases}$.

Line Integral Of Vector Field

- Let C be a smooth curve in $\mathbb{R}^n$, and $\boldsymbol{r}(t)$ is a bijective smooth parametrisation of C. Let $\boldsymbol{F}: C \to \mathbb{R}^n$ be a continuous vector field on C. Then $\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_I \boldsymbol{F}\big(\boldsymbol{r}(t)\big)\cdot \boldsymbol{r}'(t)dt$.

  - This depends on the chosen $\boldsymbol{r}$ up to orientation.
- **Fundamental Theorem of Line Integrals**: When C is smooth/piecewise smooth, $\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{B}) - f(\boldsymbol{A})$.

6. Surface Integral

Surface Integral of Scalar-Valued Function

- Let S be a smooth surface in $\mathbb{R}^n$, and $\boldsymbol{r}(u,v)$ is a bijective smooth parametrisation of S. Let $G: S \to \mathbb{R}$ be a scalar-valued function on S. Then $\int_S G\, d\sigma = \int_R G(\boldsymbol{r}(u,v))|\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)|d(u,v)$.
  - **Area of Smooth Surface**: $\int_S 1\, d\sigma = \int_R |\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)|d(u,v)$.
- **Additivity Rule**: If a piecewise smooth surface S is made up of finite number of smooth curves $S_1, S_2, \dots, S_n$, then $\int_S f\, ds = \int_{S_1} f\, ds + \int_{S_2} f\, ds + \cdots + \int_{S_n} f\, ds$.

Surface Integral of Vector Field

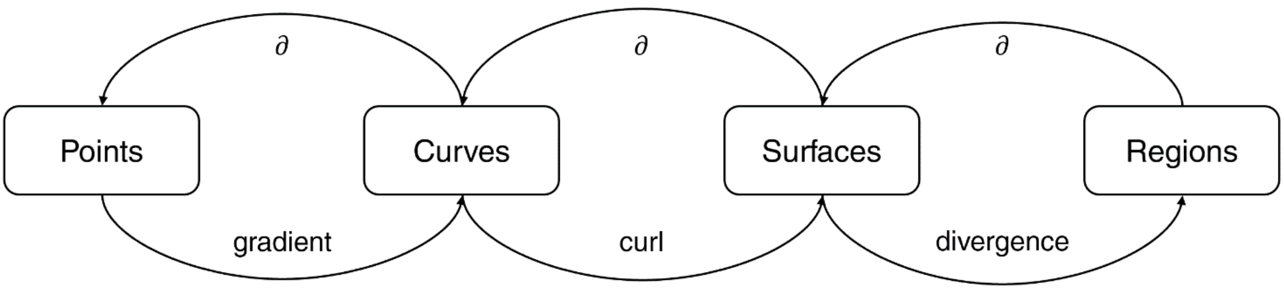
- Let S be a smooth surface in $\mathbb{R}^n$, and $\boldsymbol{r}(u,v)$ is a bijective smooth parametrisation of S. Let $\boldsymbol{F}: S \to \mathbb{R}^3$ be a continuous vector field on S. Then $\int_S \boldsymbol{F}\, d\boldsymbol{S} = \int_S \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma = \int_R \boldsymbol{F}(\boldsymbol{r}(u,v)) \cdot \boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)\, d(u,v)$.
  - This depends on the chosen $\boldsymbol{r}$ up to orientation.

Curl and Divergence

- Let U be an open set in $\mathbb{R}^3$, $\boldsymbol{F}: U \to \mathbb{R}^3$ be a differentiable vector field. Then the **curl** of $\boldsymbol{F}$ is the vector field $\nabla \times \boldsymbol{F}: U \to \mathbb{R}^3$ given by $(\nabla \times \boldsymbol{F})(\boldsymbol{p}) = \begin{pmatrix} \frac{\partial P}{\partial y}(\boldsymbol{p}) - \frac{\partial N}{\partial z}(\boldsymbol{p}) \\ \frac{\partial M}{\partial z}(\boldsymbol{p}) - \frac{\partial P}{\partial x}(\boldsymbol{p}) \\ \frac{\partial N}{\partial x}(\boldsymbol{p}) - \frac{\partial M}{\partial y}(\boldsymbol{p}) \end{pmatrix}$.
  - $\nabla \times \boldsymbol{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} M \\ N \\ P \end{pmatrix}$.
  - $\nabla \times \boldsymbol{F}$ is continuous vector field if and only if $\boldsymbol{F}$ is continuously differentiable.
  - Let $\boldsymbol{G} = \nabla \times \boldsymbol{F}$, then $\boldsymbol{G}$ is a curl vector field and $\boldsymbol{F}$ is a vector potential of $\boldsymbol{G}$.
- Sum Rule, Difference Rule, Constant Multiple Rule and Product Rule apply.
  - **Product Rule**: $\nabla \times (f\boldsymbol{F}) = f(\nabla \times \boldsymbol{F}) + (\nabla f) \times \boldsymbol{F}$.
- Given f is twice continuously differentiable, $\nabla \times (\nabla f) = \boldsymbol{0}$.
- **Stokes' Theorem**: Let S be a smooth surface in $\mathbb{R}^3$, and $\boldsymbol{r}(u,v)$ is a bijective smooth parametrisation of S. Let ∂S be the counter-clockwise boundary of S. Let $\boldsymbol{F}$ be a continuously differentiable vector field defined on S, then $\int_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, d\sigma = \oint_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\partial S} \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t) dt$.
  - **Green's Theorem**: $\oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$.
- Let U be an open set in $\mathbb{R}^3$, $\boldsymbol{F}: U \to \mathbb{R}^3$ be a differentiable vector field. Then the **divergence** of $\boldsymbol{F}$ is the scalar valued function $\nabla \cdot \boldsymbol{F}: U \to \mathbb{R}$ given by $(\nabla \cdot \boldsymbol{F})(\boldsymbol{p}) = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$.

- $\nabla \cdot \boldsymbol{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} M \\ N \\ P \end{pmatrix}$.
- $\nabla \cdot \boldsymbol{F}$ is continuous if $\boldsymbol{F}$ is continuously differentiable.

- Sum Rule, Difference Rule, Constant Multiple Rule and Product Rule apply.
  - **Product Rule**: $\nabla \cdot (f\boldsymbol{F}) = f(\nabla \cdot \boldsymbol{F}) + (\nabla f) \cdot \boldsymbol{F}$.
- Given $\boldsymbol{F}$ is twice continuously differentiable, $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$.
- **Divergence Theorem**: Let D be a nice region in $\mathbb{R}^3$, whose boundary ∂D is a piecewise smooth surface. Let $\boldsymbol{F}: D \to \mathbb{R}^3$ be a continuously differentiable vector field on D. Then $\int_D (\nabla \cdot \boldsymbol{F}) dV = \oint_{\partial D} \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma$.



<u>**Fundamental Theorem**</u>

$$\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{B}) - f(\boldsymbol{A})$$

<u>**Stokes' Theorem**</u>

$$\int_S (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n}\, d\sigma = \oint_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r}$$

<u>**Divergence Theorem**</u>

$$\int_D (\nabla \cdot \boldsymbol{F}) dV = \oint_{\partial D} \boldsymbol{F} \cdot \boldsymbol{n}\, d\sigma$$

7. Application of Multivariable Calculus

<u>Extreme Value</u>

- Let $f: R \to \mathbb{R}$ be a scalar-valued function. If R is compact and f is continuous, then there exists a **global maximum** value $f(\boldsymbol{p})$ such that $\forall \boldsymbol{q} \in R, f(\boldsymbol{q}) \le f(\boldsymbol{p})$.
- $\boldsymbol{p} \in R$ is a **local maximum** point for f if and only if $\exists \epsilon \in \mathbb{R}_{\ge 0}$ such that $\forall \boldsymbol{q} \in R$ with $|q - p| < \epsilon$, one has $f(\boldsymbol{q}) \le f(\boldsymbol{p})$.
- **First Derivative Test**: If $\boldsymbol{p}$ is a local maximum/minimum point of f, then $(\nabla f)(\boldsymbol{p}) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\boldsymbol{p}) \\ \cdots \\ \frac{\partial f}{\partial x_n}(\boldsymbol{p}) \end{pmatrix} = \boldsymbol{0}$.
- $\boldsymbol{p} \in R$ is a **critical point** of f when all partial derivatives of f is 0 (or some does not exist).
- $\boldsymbol{p} \in R$ is a **saddle point** if and only if it is a critical point but not a local maximum/minimum point.
- **Second Derivative Test**: Assume f os twice continuously differentiable. The **Hessian** of f is the matrix-valued function given by $H_f(\boldsymbol{p}) = [\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{p})]$. Suppose $\boldsymbol{p}$ is a critical point of f, then:
  - If $H_f(\boldsymbol{p})$ is negative definite (all eigenvalues are negative), then $\boldsymbol{p}$ is a local maximum point.
  - If $H_f(\boldsymbol{p})$ is positive definite (all eigenvalues are positive), then $\boldsymbol{p}$ is a local minimum point.
  - If $H_f(\boldsymbol{p})$ is indefinite (some eigenvalues are opposite signed), then $\boldsymbol{p}$ is a saddle point.
  - Otherwise, the test is inconclusive.
  - What does this imply in $\mathbb{R}^2$?

Maxwell's Equations

Gauss's Law for Electric Fields

$$\Phi_E = \oint_S \boldsymbol{E} \cdot \boldsymbol{n} \, dA = \frac{q_{enc}}{\varepsilon_0}$$

- Interpretation: The flux of an electric field passing through any closed surface is proportional to the total charge contained within that surface.
- Φ_E: Electric flux through surface S.
- S: Closed surface.
- $\boldsymbol{E}$: Electric field (electrical force per unit charge).
- q_{enc}: Total amount of charge contained within surface S.
- ε_0: Electric constant.

Apply Divergence Theorem to L.H.S.:

$$LHS = \int_V \nabla \cdot \boldsymbol{E} \, dV$$

Rewrite q_{enc} in terms of electric charge density:

$$RHS = \int_V \frac{\rho}{\varepsilon_0} dV$$

Gauss's Law for Electric Fields

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$

- $\boldsymbol{E}$: Electric field (electrical force per unit charge).
- ρ: Electric charge density.
- ε_0: Electric constant.

Gauss's Law for Magnetic Fields

$$\Phi_B = \oint_S \boldsymbol{B} \cdot \boldsymbol{n} \, dA = 0$$

- Interpretation: The total magnetic flux passing through any closed surface is zero.
- Φ_B: Magnetic flux through surface S.
- S: Closed surface.
- $\boldsymbol{B}$: Magnetic field.

Apply Divergence Theorem to L.H.S.:

$$LHS = \int_V \nabla \cdot \boldsymbol{B} \, dV$$

$$RHS = 0$$

Gauss's Law for Magnetic Fields

$$\nabla \cdot \boldsymbol{B} = 0$$

- $\boldsymbol{B}$: Magnetic field.

Faraday's Law

$$\oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{d}{dt}\int_S \boldsymbol{B} \cdot \boldsymbol{n}\, dA$$

- Interpretation: Changing magnetic flux through a surface induces an e.m.f. in any boundary path of that surface and a changing magnetic field induces a circulating electric field.
- $e.m.f. = \oint_C \boldsymbol{E} \cdot d\boldsymbol{l}$: Electromotive force around path C.
- C: Boundary of surface S.
- $\boldsymbol{E}$: Induced electric fields along path C.
- $\Phi_B = \int_S \boldsymbol{B} \cdot \boldsymbol{n}\, dA$: Magnetic flux through surface S.
- S: Any surface (not necessarily closed).
- $\boldsymbol{B}$: Magnetic field.

Apply Stokes' Theorem to L.H.S.:

$LHS = \int_S (\nabla \times \boldsymbol{E}\,) \cdot \boldsymbol{n}\, dA$

Put differentiation in R.H.S. under integral sign:

$RHS = \int_S -\frac{\partial \boldsymbol{B}}{\partial t} \cdot \boldsymbol{n}\, dA$

↓

Faraday's Law

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

- $\boldsymbol{E}$: Electric field (electrical force per unit charge).
- $\boldsymbol{B}$: Magnetic field.

Ampère-Maxwell Law

$$\oint_C \boldsymbol{B} \cdot d\boldsymbol{l} = \mu_0\left(I_{enc} + \varepsilon_0 \frac{d}{dt}\int_S \boldsymbol{E} \cdot \boldsymbol{n}\, dA\right)$$

- $\oint_C \boldsymbol{B} \cdot d\boldsymbol{l}$: Magnetic flux circulation around path C.
- C: Boundary of surface S.
- $\boldsymbol{B}$: Induced magnetic field.
- μ_0: Permeability of free space.
- I_{enc}: "Enclosed current", the net current that penetrates surface S.
- ε_0: Permittivity of free space.
- S: Any surface (usually not closed).
- $\boldsymbol{E}$: Electric field.
- $\Phi_E = \int_S \boldsymbol{E} \cdot \boldsymbol{n}\, dA$: Electric flux through surface S.

Apply Stokes' Theorem to L.H.S.:

$LHS = \int_S (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{n}\, dV$

Write I_{enc} in terms of current density and put differentiation inside integral sign:

$RHS = \int_S \mu_0 (\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}) \cdot \boldsymbol{n}\, dA$

↓

Ampère-Maxwell Law

$$\nabla \times \boldsymbol{B} = \mu_0\left(\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

- $\boldsymbol{B}$: Induced magnetic field.
- μ_0: Permeability of free space.
- $\boldsymbol{J}$: Electric current density.
- ε_0: Permittivity of free space.
- $\boldsymbol{E}$: Electric field.

- **Laplacian of Scalar-Valued Function**: Divergence of gradient, $\nabla^2 f = \nabla \cdot (\nabla f)$.
- **Laplacian of Vector Field**: Gradient of divergence minus curl of curl, $\nabla^2 \boldsymbol{A} = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla \times (\nabla \times \boldsymbol{A})$.

References

AY2020/21 Semester 2 MA2104 Lecture Notes by Prof. Chin Chee Whye.