MA2108 Mathematical Analysis I

AY2021/22 Semester 1

CHAPTER 1 – PRELIMINARIES

1.1. Sets and Functions

1.1.9. (*Definition*) Let $f: A \to B$ be a function from A to B.
a. The function f is said to be injective if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.
b. The function f is said to be surjective if $f(A) = B$.
c. If f is both injective and surjective, then f is said to be bijective.

1.1.11. (*Definition*) Let $f: A \to B$ be a bijection of A onto B. Then the inverse function $f^{-1}: B \to A$ is defined such that $f^{-1}(f(x)) = x, \forall x \in A$ and $f(f^{-1}(y)) = y, \forall y \in B$.

1.2. Mathematical Induction

1.2.1. (*Well-Ordering Property of* $\mathbb{N}$) Every non-empty subset S of $\mathbb{N}$ has a least element, i.e. there exists $m \in S$ such that $m \leq k$ for all $k \in S$.

1.3. Finite and Infinite Sets

1.3.8. (*Theorem*) The set $\mathbb{N} \times \mathbb{N}$ is countably infinite.

1.3.11. (*Theorem*) The set $\mathbb{Q}$ of all rational numbers is denumerable.

CHAPTER 2 – THE REAL NUMBERS

2.1. The Algebraic and Order Properties of $\mathbb{R}$

2.1.1. (*Axioms of the Algebraic Properties of* $\mathbb{R}$)
(A1) (*Commutative Property of Addition*) $\forall a, b \in \mathbb{R}\ \{a + b = b + a\}$.
(A2) (*Associative Property of Addition*) $\forall a, b, c \in \mathbb{R}\ \{(a + b) + c = a + (b + c)\}$.
(A3) (*Existence of Additive Identity*) $\exists 0 \in \mathbb{R}, \forall a \in \mathbb{R}\ \{0 + a = a + 0 = a\}$.
(A4) (*Existence of Additive Inverse*) $\forall a \in \mathbb{R}, \exists(-a) \in \mathbb{R}\ \{a + (-a) = (-a) + a = 0\}$.
(M1) (*Commutative Property of Multiplication*) $\forall a, b \in \mathbb{R}\ \{a \cdot b = b \cdot a\}$.
(M2) (*Associative Property of Multiplication*) $\forall a, b, c \in \mathbb{R}\ \{(a \cdot b) \cdot c = a \cdot (b \cdot c)\}$.
(M3) (*Existence of Multiplicative Identity*) $\exists 1 \in \mathbb{R}, \forall a \in \mathbb{R}\ \{1 \cdot a = a \cdot 1 = a\}$.
(M4) (*Existence of Multiplicative Inverse*) $\forall a \neq 0 \in \mathbb{R}, \exists \frac{1}{a} \in \mathbb{R}\ \{a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1\}$.
(D) (*Distributive Property of Multiplication over Addition*)

$\forall a, b, c \in \mathbb{R} \{a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a\}$.

2.1.4. (*Theorem*) There does not exist a rational number *r* such that $r^2 = 2$.

<u>*Proof*</u>

Assume $\exists r \in \mathbb{Q} \{r^2 = 2\}$.

Then $p, q \in \mathbb{Z}^+ \left\{\left(r = \frac{p}{q}\right) \wedge (\gcd(p, q) = 1)\right\}$.

Since $r^2 = 2$, $p^2 = 2q^2$.

Hence p is even, let $p = 2k$ where $k \in \mathbb{Z}^+$.

Then $p^2 = 4k^2$, $q^2 = 2k^2$.

Hence q is even, which means 2 is a common factor of p and q.

This contradicts the assumption that $\gcd(p, q) = 1$, hence the assumption is false.

Hence there does not exist a rational number r such that $r^2 = 2$.

2.1.5. (*Axioms of the Order Properties of* $\mathbb{R}$) Assuming $a, b \in \mathbb{R}$:

(a) If a and b are positive, then $a + b$ is positive.

(b) If a and b are positive, then ab is positive.

(c) (The Trichotomy Property) Exactly one of the following properties holds:

a is positive, $a = 0$, or $-a$ is positive.

2.1.6. (*Definition*) Assuming $a, b \in \mathbb{R}$:

(a) If $a - b$ is positive, then we write $a > b$ or $b < a$.

(b) If $a - b$ is positive or 0, then we write $a \geq b$ or $b \leq a$.

2.1.7. (*Theorem*) Assuming $a, b, c \in \mathbb{R}$:

(a) $(a < b) \wedge (b < c) \Rightarrow (a < c)$.

(b) $(a < b) \Rightarrow (a + c < b + c)$.

(c) $(a < b) \wedge (c > 0) \Rightarrow ac < bc$ and $(a < b) \wedge (c < 0) \Rightarrow ac > bc$.

2.1.8. (*Theorem*)

(a) $\forall a \in \mathbb{R}\backslash\{0\} \{a^2 > 0\}$.

(b) $1 > 0$.

(c) $\forall a \in \mathbb{N}\backslash\{0\} \{a > 0\}$.

<u>*Proof*</u>

(a) Since $a \in \mathbb{R}\backslash\{0\}$, by The Trichotomy Property, $a > 0$ or $-a > 0$.

If $a > 0$, then $a^2 = a \cdot a > 0$. (By Axioms 2.1.5b)

If $a < 0$, then $a^2 = (-a) \cdot (-a) > 0$. (By Axioms 2.1.5b)

(b) Since $1 = 1^2$, $1 > 0$. (By Theorem 2.1.8a)

2.1.9. (*Theorem*) If $a \in \mathbb{R}$ satisfies $\forall \epsilon > 0 \{0 \leq a < \epsilon\}$, then $a = 0$.

<u>*Proof*</u>

Suppose $a > 0$.

Choose $\epsilon = \frac{a}{2}$, then $0 < \epsilon < a$, leading to a contradiction.
Hence $a = 0$.

2.1.10. (*Theorem*) If $ab > 0$, then either $a > 0, b > 0$ or $a < 0, b < 0$.

2.1.11. (*Corollary*) If $ab < 0$, then either $a > 0, b < 0$ or $a < 0, b > 0$.

2.2. Absolute Value and the Real Line

2.2.1. (*Definition*) Suppose $a \in \mathbb{R}$. The absolute value of a is defined by

$$|a| = \begin{cases} a & (a > 0) \\ 0 & (a = 0). \\ -a & (a < 0) \end{cases}$$

2.2.2. (*Theorem of Properties of Absolute Value*)

(a) $\forall a, b \in \mathbb{R}\,\{|ab| = |a||b|\}$.

(b) $\forall a \in \mathbb{R}\,\{|a|^2 = a^2\}$.

(c) If $c \geq 0$, then $|a| \leq c \Rightarrow -c \leq a \leq c$.

(d) $\forall a \in \mathbb{R}\,\{-|a| \leq a \leq |a|\}$.

2.2.3. (*Theorem of Triangle Inequality*) $\forall a, b \in \mathbb{R}\,\{|a + b| \leq |a| + |b|\}$.

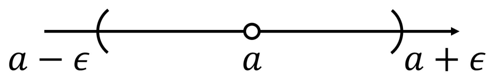
2.2.4. (*Corollary*)

(a) $\forall a, b \in \mathbb{R}\,\{\big||a| - |b|\big| \leq |a - b|\}$.

(b) $\forall a, b \in \mathbb{R}\,\{|a - b| \leq |a| + |b|\}$.

2.2.5. (*Corollary*) $\forall a_1, a_2, \dots, a_n \in \mathbb{R}\,\{|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|\}$.

2.2.7. (*Definition*) Let $a \in \mathbb{R}$ and $\epsilon > 0$. Then the ϵ-neighbourhood of a is the set $V_\epsilon(a) \coloneqq \{x \in \mathbb{R} : |x - a| < \epsilon\}$.



2.2.8. (*Theorem*) Let $a \in \mathbb{R}$. If x belongs to $V_\epsilon(a)$ for every $\epsilon > 0$, then $x = a$.

2.3. The Completeness Property of ℝ

2.3.1. (*Definition*) Let S be a non-empty subset of $\mathbb{R}$.

(a) A number u is called an upper bound of S if $\forall s \in S\,\{s \leq u\}$. If such u exists, S is bounded above.

(b) A number w is called a lower bound of S if $\forall s \in S \{w \leq s\}$. If such w exists, S is bounded below.

(c) A set is bounded if it is both bounded above and bounded below, otherwise it is unbounded.

2.3.2. (*Definition*) Let S be a non-empty subset of $\mathbb{R}$.

(a) A number u is called a supremum of S if it satisfies the following conditions:

(1) u is an upper bound of S;

(2) If v is an upper bound of S, then $u \leq v$.

(b) A number w is called a infimum of S if it satisfies the following conditions:

(1) w is a lower bound of S;

(2) If v is a lower bound of S, then $w \geq v$.

2.3.3. (*Lemma / Equivalent Definition*) Let S be a non-empty subset of $\mathbb{R}$.

(a) A number u is called a supremum of S if it satisfies the following conditions:

(1) u is an upper bound of S;

(2) If $v < u$, then $\exists s' \in S \{v < s'\}$.

(b) A number w is called a infimum of S if it satisfies the following conditions:

(1) w is a lower bound of S;

(2) If $v > w$, then $\exists s' \in S \{v > s'\}$.

2.3.4. (*Lemma*) Let u be an upper bound of $S \subset \mathbb{R}$. Then $u = \sup S$ if and only if $\forall \epsilon > 0, \exists S_\epsilon \in S \{u - \epsilon < S_\epsilon\}$.

2.3.6. (*Axioms of Supremum Property of* $\mathbb{R}$) Every non-empty subset of $\mathbb{R}$ that has an upper bound has a supremum.

(*Axioms of Infimum Property of* $\mathbb{R}$) Every non-empty subset of $\mathbb{R}$ that has a lower bound has an infimum.

2.4. Applications of the Supremum Property

2.4.3. (*Archimedean Property*) If $x \in \mathbb{R}$, then $\exists n_x \in \mathbb{N} \{x \leq n_x\}$.

Proof

Suppose $\exists x \in \mathbb{R}, \forall n \in \mathbb{N} \{x > n\}$.

Then x is an upper bound of $\mathbb{N}$.

By Supremum Property, $\mathbb{N}$ has a supremum u.

Since $u = \sup \mathbb{N}$, $\exists n \in \mathbb{N} \{u - 1 < n\}$. (By Lemma 2.3.3a)

Then $u < n + 1$.

Since $n + 1 \in \mathbb{N}$, u is not an upper bound, and therefore not a supremum.

Therefore, $\forall x \in \mathbb{R}, \exists n \in \mathbb{N} \{x > n\}$.

2.4.4. (*Corollary*) Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$, then $\inf S = 0$.

2.4.5. (*Corollary*) $\forall \epsilon > 0, \exists n \in \mathbb{N} \{\frac{1}{n} < \epsilon\}$.

2.4.6. (*Corollary*) If $x > 0$, then $\exists n \in \mathbb{N} \{n - 1 < x < n\}$.

2.4.7. (*Theorem*) There exists a unique positive real number b such that $b^2 = 2$.

Proof

1. Existence
    - 1.1. Existence of $\sup S$, where $S = \{x \in \mathbb{R} : (x > 0) \wedge (x^2 < 2)\}$
        - 1.1.1. Since $1 \in S$, $S \neq \emptyset$.
        - 1.1.2. Suppose $x > 2$, then $x^2 > 4$, hence $x \notin S$.
        - 1.1.3. Hence $(x \in S) \Rightarrow (x \leq 2)$.
        - 1.1.4. Hence 2 is an upper bound of S, S is bounded above.
        - 1.1.5. By Supremum Property, $\sup S$ exists.
    - 1.2. Existence of positive real number b such that $b^2 = 2$
        - 1.2.1. Let $b = \sup S$.
        - 1.2.2. Suppose $b^2 < 2$.
            - 1.2.2.1. Then $\frac{2b+1}{2-b^2} > 0$.
            - 1.2.2.2. By Archimedean Property, $\exists n \in \mathbb{N} \left\{\frac{2b+1}{2-b^2} \leq n\right\}$.
            - 1.2.2.3. $\left(b + \frac{1}{n}\right)^2 = b^2 + \frac{2b}{n} + \frac{1}{n^2} \leq b^2 + \frac{2b+1}{n} \leq b^2 + 2 - b^2 = 2$.
            - 1.2.2.4. Hence $b + \frac{1}{n} \in S$.
            - 1.2.2.5. Since $b + \frac{1}{n} > b$, this contradicts 1.2.1. Hence 1.2.2 is false.
        - 1.2.3. Suppose $b^2 > 2$.
            - 1.2.3.1. Then $\frac{2b}{b^2-2} > 0$.
            - 1.2.3.2. By Archimedean Property, $\exists n \in \mathbb{N} \left\{\frac{2b}{b^2-2} \leq n\right\}$.
            - 1.2.3.3. $\left(b - \frac{1}{n}\right)^2 = b^2 - \frac{2b}{n} + \frac{1}{n^2} > b^2 - \frac{2b}{n} \geq b^2 - b^2 + 2 = 2$.
            - 1.2.3.4. Hence $\forall x \in S \left\{x^2 < 2 < \left(b - \frac{1}{n}\right)^2\right\}$.
            - 1.2.3.5. Hence $b - \frac{1}{n}$ is an upper bound of S.
            - 1.2.3.6. Since $b - \frac{1}{n} < b$, this contradicts 1.2.1. Hence 1.2.3 is false.
        - 1.2.4. Hence $b^2 = 2$. Such b exists.
2. Uniqueness
    - 2.1. Suppose $a^2 = 2$.
    - 2.2. Suppose $a < \sup S$.
        - 2.2.1. $a^2 - 2 = a^2 - (\sup S)^2 = (a + \sup S)(a - \sup S) < 0$.
        - 2.2.2. Hence $a^2 < 2$. This contradicts 2.1, hence 2.2 is false.
    - 2.3. Suppose $a > \sup S$.
        - 2.3.1. $a^2 - 2 = a^2 - (\sup S)^2 = (a + \sup S)(a - \sup S) > 0$.

2.3.2. Hence $a^2 > 2$. This contradicts 2.1, hence 2.3 is false.

2.4. Hence $a = \sup S$. This proves its uniqueness.

2.4.8. (*The Density Theorem of* $\mathbb{Q}$) $\forall x, y \in \mathbb{R}\, \{(x < y) \wedge (\exists r \in \mathbb{Q}\, \{x < r < y\})\}$.

2.4.9. (*The Density Theorem of Irrational Numbers*) $\forall x, y \in \mathbb{R}\, \{(x < y) \wedge (\exists r \in \mathbb{R} \backslash \mathbb{Q}\, \{x < r < y\})\}$.

2.5. Intervals

2.5.1. (*Theorem of Nested Interval Property*) If $I_n = [a_n, b_n], n \in \mathbb{N}$ is a nested sequence of close bounded intervals, then $\exists \xi \in \mathbb{R}, \forall n \in \mathbb{N}\, \{\xi \in I_n\}$.

2.5.2. (*Theorem*) If $I_n = [a_n, b_n], n \in \mathbb{N}$ is a nested sequence of close bounded intervals such that $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then the number ξ contained in all intervals is unique.

CHAPTER 3 – SEQUENCES AND SERIES

3.1. Sequences and Their Limits

3.1.1. (*Definition*) A sequence in $\mathbb{R}$ is a real-valued function $X: \mathbb{N} \to \mathbb{R}$. The numbers $X(n), n = 1,2,3, \ldots$ are called terms of the sequence.

3.1.3. (*Definition*) A sequence $X = (x_n)$ in $\mathbb{R}$ is said to be convergent to $x \in \mathbb{R}$, or x is said to be a limit of (x_n) if $\forall \epsilon > 0, \exists K = K(\epsilon) \in \mathbb{N}\, \left\{\forall n \geq K(\epsilon)\, \{|x_n - x| < \epsilon\}\right\}$. If such limit exists, X is convergent; otherwise, it is divergent.

3.1.4. (*Theorem*) If (x_n) converges, then it has only one limit.

3.1.5. (*Theorem*) Let $X = (x_n)$ be sequence of real numbers and $x \in \mathbb{R}$, then the following statements are equivalent:

(a) X converges to x.

(b) $\forall \epsilon > 0, \exists K \in \mathbb{N}\, \left\{\forall n \geq K\, \{|x_n - x| < \epsilon\}\right\}$.

(c) $\forall \epsilon > 0, \exists K \in \mathbb{N}\, \left\{\forall n \geq K\, \{x - \epsilon < x_n < x + \epsilon\}\right\}$.

(d) For every ϵ-neighbourhood $V_\epsilon(x)$ of x, there exists a natural number K such that $\forall n \geq K\, \{x_n \in V_\epsilon(x)\}$.

3.2. Limit Theorems

3.2.1. (*Definition*) A sequence $X = (x_n)$ of real numbers is said to be bounded if there exists a real number $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

3.2.2. (*Theorem*) A convergent sequence of real numbers is bounded.

3.2.3. (*Theorem*) If $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $c \in \mathbb{R}$, then

(a) $\lim_{n\to\infty}(x_n + y_n) = x + y$.

(b) $\lim_{n\to\infty}(x_n - y_n) = x - y$.

(c) $\lim_{n\to\infty}(x_n \cdot y_n) = x \cdot y$.

(d) $\lim_{n\to\infty} c(x_n) = cx$.

(e) $\lim_{n\to\infty}(x_n / y_n) = x/y$, provided $\forall n \in \mathbb{N}\ \{y_n \neq 0\}$ and $y \neq 0$.

3.2.4. (*Theorem*) If $\forall n \in \mathbb{N}\ \{x_n > 0\}$ and (x_n) converges, then $\lim_{n\to\infty} x_n \geq 0$.

3.2.5. (*Theorem*) If (x_n) and (y_n) are convergent and $\forall n \in \mathbb{N}\ \{x_n \geq y_n\}$, then $\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n$.

3.2.6. (*Theorem*) If $a, b \in \mathbb{R}$ and $\forall n \in \mathbb{N}\ \{a \leq x_n \leq b\}$ and (x_n) is convergent, then $a \leq \lim_{n\to\infty} x_n \leq b$.

3.2.7. (*Squeeze Theorem*) Suppose that $X = (x_n), Y = (y_n), Z = (z_n)$ are sequences of real numbers such that $\forall n \in \mathbb{N}\ \{x_n \leq y_n \leq z_n\}$ and that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = a$, then Y is convergent and $\lim_{n\to\infty} y_n = a$.

3.3. Monotone Sequences

3.3.1. (*Definition*) We say the sequence (x_n) is increasing if $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$.

We say the sequence (x_n) is decreasing if $x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n \geq x_{n+1} \geq \cdots$.

We say the sequence (x_n) is monotone if it is either increasing or decreasing.

3.3.2. (*Monotone Convergence Theorem*) Let (x_n) be a monotone sequence of real numbers. Then (x_n) is convergent if and only if (x_n) is bounded.

Particularly, if (x_n) is bounded and increasing, $\lim_{x\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$.

If (x_n) is bounded and decreasing, $\lim_{x\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$.

3.4 Subsequences and the Bolzano-Weierstrass Theorem

3.4.1. (*Definition*) Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by $(x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots)$ is a subsequence of X.

3.4.2. (*Theorem*) If (x_n) converges to x, then any subsequence (x_{n_k}) also converges to x.

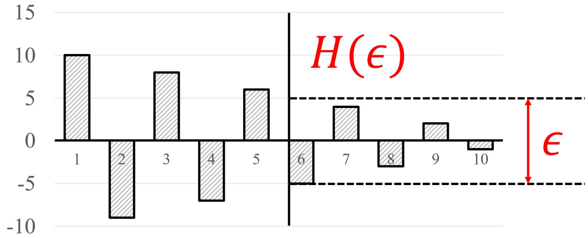
3.4.5. (*Theorem*) If (x_n) has either of the following properties, then (x_n) is divergent:
(a) (x_n) has two convergent subsequences whose limits are not equal.
(b) (x_n) is unbounded.

3.4.7. (*Theorem*) Every sequence has a monotone subsequence.

3.4.8. (*Bolzano-Weierstrass Theorem*) Every bounded sequence has a convergent subsequence.

3.5. The Cauchy Criterion

3.5.1. (*Definition*) A sequence (x_n) is said to be a Cauchy sequence if $\forall \epsilon > 0, \exists H = H(\epsilon) \in \mathbb{N}\ \forall n, m \geq H\ \{|x_n - x_m| < \epsilon\}$.



3.5.4. (*Lemma*) A Cauchy sequence of real number is bounded.

3.5.5. (*Theorem*) A sequence of real number is convergent if and only if it is a Cauchy sequence.

3.5.7. (*Definition*) A sequence $X = (x_n)$ is said to be contractive if and only if $\forall n \in \mathbb{N}, \exists 0 < C < 1\ \{|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|\}$. The number C is called the constant of the contractive sequence.

3.5.8. (*Theorem*) Every contractive sequence is a Cauchy sequence and hence is convergent.

3.6. Properly Divergent Sequences

3.6.1. (*Definition*) Let (x_n) be a sequence of real numbers. We say that (x_n) tends to $+\infty$ ($\lim_{n\to\infty} x_n = +\infty$) if $\forall\alpha \in \mathbb{R}, \exists K = K(\alpha) \in \mathbb{R} \left\{\forall n \geq K(\alpha) \left\{x_n > \alpha\right\}\right\}$.

We say that (x_n) tends to $-\infty$ ($\lim_{n\to\infty} x_n = -\infty$) if $\forall\beta \in \mathbb{R}, \exists K = K(\beta) \in \mathbb{R} \left\{\forall n \geq K(\alpha\beta) \left\{x_n < \beta\right\}\right\}$.

We say that (x_n) is properly divergent if either $\lim_{n\to\infty} x_n = +\infty$ or $\lim_{n\to\infty} x_n = -\infty$.

3.6.2. (*Theorem*) Let (x_n) and (y_n) be two sequences of real numbers and suppose that $\forall n \in \mathbb{N} \left\{x_n \leq y_n\right\}$, then:

(a) If $\lim_{n\to\infty} x_n = +\infty$, then $\lim_{n\to\infty} y_n = +\infty$.

(b) If $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.

3.6.3. (*Theorem*) If (x_n) is an unbounded increasing sequence, then $\lim_{n\to\infty} x_n = +\infty$. If (x_n) is an unbounded decreasing sequence, then $\lim_{n\to\infty} x_n = -\infty$.

3.7. Introduction to Infinite Series

3.7.1. (*Definition*) Let $X = (x_n)$ be a sequence of real numbers, then the infinite series generated by X is the sequence $S = (s_k)$ defined by:

$$s_1 = x_1$$
$$s_2 = s_1 + x_2$$
$$\dots$$
$$s_k = s_{k-1} + x_k$$
$$\dots$$

The numbers x_n are called the terms of the series and the numbers s_k are called the partial sums of the series.

If $\lim_{n\to\infty} s_n$ exists, we say that S is convergent and $\lim_{n\to\infty} s_n$ is called the sum or value of the series; otherwise, S is divergent.

Convergence Tests

3.7.3. (*Theorem of the n-th term test*) If the series $\sum x_n$ converges, then $\lim_{n\to\infty} x_n = 0$. Or equivalently, if $\lim_{n\to\infty} x_n \neq 0$, the series $\sum x_n$ diverges.

3.7.4. (*Theorem of Cauchy-criterion test*) The series $\sum x_n$ converges if and only if $\forall\epsilon > 0, \exists M = M(\epsilon) \in \mathbb{N} \left\{\forall m > n > M(\epsilon) \left\{|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \epsilon\right\}\right\}$.

3.7.5. (*Theorem of partial sum bounded test for series with non-negative terms*) Suppose $\forall n \in \mathbb{N}\ \{x_n \geq 0\}$. Then the series $\sum x_n$ converges if and only if the sequence (s_n) of partial sums is bounded.

3.7.7. (*Comparison Test*) Let $(x_n), (y_n)$ be real sequences and suppose that for some $K \in \mathbb{N}$, we have $\forall n \geq K\ \{0 \leq x_n \leq y_n\}$. Then:

(a) The convergence of $\sum y_n$ implies the convergence of $\sum x_n$.
(b) The divergence of $\sum x_n$ implies the divergence of $\sum y_n$.

3.7.8. (*Limit Comparison Test*) Let $(x_n), (y_n)$ be strictly positive sequences and suppose that the following limit exists:

$$r = \lim_{n\to\infty} \frac{x_n}{y_n}$$

(a) If $r > 0$, then $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent.
(b) If $r = 0$ and if $\sum y_n$ is convergent, then $\sum x_n$ is convergent.

CHAPTER 9 – INFINITE SERIES

9.1. Absolute Convergence

9.1.1. (*Definition*) The series $\sum x_n$ is absolutely convergent if the series $\sum |x_n|$ is convergent.

A series is said to be conditionally convergent if it is convergent but it is not absolutely convergent.

9.1.2. (*Theorem*) If a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, then it is convergent.

9.2. Tests for Absolute Convergence

9.2.1. (*Limit Comparison Test II*) Suppose that (x_n), (y_n) are non-zero sequences and suppose that the following limits exists:

$$r \coloneqq \lim_{n\to\infty} \left(\frac{|x_n|}{|y_n|}\right)$$

a. If $r > 0$, then $\sum x_n$ is absolutely convergent if and only if $\sum y_n$ ia absolutely convergent.
b. If $r = 0$, then if $\sum y_n$ is absolutely convergent, then $\sum x_n$ is absolutely convergent.

9.2.2. (*Root Test*) Let (x_n) be a sequence.

a. If there exist $r \in \mathbb{R}$ with $0 \leq r < 1$ and $K \in \mathbb{N}$ such that $|x_n|^{\frac{1}{n}} \leq r$ for $n \geq K$, then the series $\sum x_n$ is absolutely convergent.

b. If there exists $K \in \mathbb{N}$ such that $|x_n|^{\frac{1}{n}} \geq 1$ for $n \geq K$, then the series $\sum x_n$ is divergent.

9.2.3. (*Corollary of another version of root test*) Suppose that the limit $r \coloneqq \lim_{n\to\infty} |x_n|^{\frac{1}{n}}$ exists. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.

9.2.4. (*Ratio Test*) Let (x_n) be a sequence of nonzero real numbers.

a. If there exist r with $0 < r < 1$ and $K \in \mathbb{N}$ such that $\left|\frac{x_{n+1}}{x_n}\right| \leq r$ for $n \geq K$, then $\sum x_n$ is absolutely convergent.

b. If there exists $K \in \mathbb{N}$ such that $\left|\frac{x_{n+1}}{x_n}\right| \geq 1$ for $n \geq K$, then $\sum x_n$ is divergent.

9.2.5. (*Corollary of another version of ratio test*) Let (x_n) be a sequence of nonzero real numbers and suppose that the limit $r \coloneqq \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ exists. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.

CHAPTER 4 – LIMITS

4.1. Limits of Functions

4.1.1. (*Definition*) Let A be a subset of $\mathbb{R}$. A point c is called a cluster point of A if for every $\delta > 0$ there exists at least one point $x \in A$ such that $0 < |x - c| < \delta$, i.e. $(V_\delta(c) \backslash \{c\}) \cap A \neq \emptyset$ for any $\delta > 0$.

4.1.2. (*Theorem of alternative definition of cluster points*) A real number c is a cluster point of A if and only if there exists a sequence (a_n) in A such that $\lim a_n = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.

4.1.4. (*Definition*) Let $A \subseteq \mathbb{R}$ and c be a cluster point of A. For a function $f: A \to \mathbb{R}$, a real number L is said to be a limit of f at c if for any given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$, that is,

$$x \in A \cap (V\_\delta\ (c)\{c\}) \Rightarrow f(x) \in V_\epsilon(L)$$

In this case, we write $\lim_{x\to c} f(x) = L$.

<u>*Example*</u>

Prove $\lim_{x\to 2} x^2 = 4$.

For any $\epsilon > 0$ we choose $0 < \delta < \min\{2 - \sqrt{4-\epsilon}, \sqrt{4+\epsilon} - 2\}$.

Then whenever $0 < |x - 2| < \delta$, we have $x \in (\sqrt{4-\epsilon}, \sqrt{4+\epsilon})$ and hence $|x^2 - 4| < \epsilon$.

4.1.5. (*Theorem of uniqueness of limit*) If $f: A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.

4.1.8. (*Sequential Criterion of Limits*) Let $f: A \to \mathbb{R}$ and a be a cluster point of A. The following statements are equivalent:

1. $\lim_{x \to a} f(x) = L$.
2. For every sequence (x_n) in A that converges to a such that $x_n \neq a$ for all n, the sequence $(f(x_n))$ converges to L.

4.2. Limit Theorems

4.2.1. (*Definition*) Let $f: A \to \mathbb{R}$ and c be a cluster point of A. We say that f is bounded on a neighbourhood of c if there exists $V_\delta(c)$ and a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A \cap V_\delta(c)$.

4.2.2. (*Theorem*) If $f: A \to \mathbb{R}$ has a limit at a cluster point c, then f is bounded on some neighbourhood of c.

4.2.3. (*Theorem*) Suppose that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Let $b \in \mathbb{R}$.

a. $\lim_{x \to c} (f \pm g)(x) = L \pm M$;

b. $\lim_{x \to c} (fg)(x) = LM$, $\lim_{x \to c} (bf)(x) = bL$;

c. If $h(x) \neq 0$ for all $x \in A$ and $\lim_{x \to c} h(x) = H \neq 0$, then $\lim_{x \to c} (\frac{f}{h})(x) = \frac{L}{H}$.

4.2.6. (*Theorem*) If $f(x) \leq g(x)$ for all $x \in A$ and both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$.

4.2.7. (*Squeeze Theorem*) Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in A$ and $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) = L$.

4.2.9. (*Theorem*) If $\lim_{x \to c} f(x) > 0$, then there exists $V_\delta(c)$ of c such that $f(x) > 0$ for all $x \in A \cap V_\delta(c)$, $x \neq c$.

4.3. Some Extensions of the Limit Concept

4.3.1. (*Definition*) Let c be a cluster point of $A \cap (c, \infty)$. We say that L is the right-hand limit of f at c if for any $\epsilon > 0$, $\exists \delta > 0$ such that $0 < x - c < \delta \Rightarrow |f(x) - L| < \epsilon$. In this case we write $\lim_{x \to c^+} f(x) = L$.

Let c be a cluster point of $A \cap (c, \infty)$. We say that L is the left-hand limit of f at c if for any $\epsilon > 0$, $\exists \delta > 0$ such that $0 < c - x < \delta \Rightarrow |f(x) - L| < \epsilon$. In this case we write $\lim_{x \to c^-} f(x) = L$.

4.3.3. (*Theorem*) $\lim_{x \to c} f(x) = L$ exists if and only if both $\lim_{x \to c^+} f(x)$ and $\lim_{x \to c^-} f(x)$ exist and $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$.

CHAPTER 5 – CONTINUOUS FUNCTIONS

5.1. Continuous Functions

5.1.1. (*ϵ-δ Definition of Continuity*) Let $A \subset \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in A$. We say that f is continuous at c if given any number $\epsilon > 0$, there exists $\delta > 0$ such that if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Equivalently, if c is a cluster point, $f(x)$ is continuous at c if and only if $f(c) = \lim_{x \to c} f(x)$.

5.1.2. (*Equivalent Definition of Continuity*) Let $A \subset \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in A$. We say that f is continuous at c if given any ϵ-neighbourhood $V_\epsilon(f(c))$ of $f(c)$, there exists a δ-neighbourhood $V_\delta(c)$ of c such that if x is any point of $A \cap V_\delta(c)$, then $f(x)$ belongs to $V_\epsilon(f(c))$, that is $f\big(A \cap V_\delta(c)\big) \subseteq V_\epsilon(f(c))$.

If f fails to be continuous at c, then we say that f is discontinuous at c.
If f is continuous at every point in A, then we say that f is continuous on A.

5.1.3. (*Sequential Criterion for Continuity*) f is continuous at $x = a$ if and only if for every sequence (x_n) in the domain of f such that $x_n \to a$, we have $f(x_n) \to f(a)$.

5.1.4. (*Discontinuity Criterion*) f is discontinuous at $x = a$ if and only if there exists a sequence (x_n) in the domain of f such that $x_n \to a$, but $f(x_n) \nrightarrow f(a)$.

5.2. Combinations of Continuous Functions

5.2.1. (*Theorem*) Suppose that f and g are continuous at $x = c$, then

a. $f \pm g$, $f \cdot g$ and bf are also continuous at $x = c$, where b is a constant.
b. If $g(c) \neq 0$, then f/g is also continuous at $x = c$.

5.2.2. (*Theorem*) Suppose that f and g are continuous on A, then

a. $f \pm g$, $f \cdot g$ and bf are also continuous on A, where b is a constant.

b. If $g(c) \neq 0$, then f/g is also continuous on A.

5.2.6. (*Theorem*) Let $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ and $f(A) \subseteq B$. If f is continuous at c, and g is continuous at $b = f(c)$, then $g \circ f$ is continuous at c.

5.2.7. (*Theorem*) Let $f: A -> \mathbb{R}$, $g: B \to \mathbb{R}$ and $f(A) \subseteq B$. If f is continuous on A and g is continuous on B, then $g \circ f$ is continuous on A.

5.3. Continuous Functions on Intervals

5.3.1. (*Definition*) A function $f: A \to \mathbb{R}$ is said to be bounded on A if there exists $M > 0$ such that $|f(x)| \leq M, \forall x \in A$.

5.3.2. (*Boundness Theorem*) If f is continuous on $[a, b]$, then f is bounded on $[a, b]$.

5.3.3. (*Definition 5.3.3*) We say that f has an absolute maximum on A if there exists $x^* \in A$ such that $f(x^*) \geq f(x), \forall x \in A$. So, in this case, $f(x^*) = \sup f(A) = \max f(A)$.

We say that f has an absolute minimum on A if there exists $x^* \in A$ such that $f(x^*) \leq f(x), \forall x \in A$. So, in this case, $f(x^*) = \inf f(A) = \min f(A)$.

5.3.4. (*Maximum-Minimum Theorem*) If f is continuous on $[a, b]$, then f has an absolute maximum and an absolute minimum on $[a, b]$.

5.3.5. (*Location of Roots Theorem*) If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, then there exists a point c in (a, b) such that $f(c) = 0$.

5.3.7. (*Intermediate Value Theorem*) Let I be an interval, f be continuous on I, and $a, b \in I$ with $f(a) \leq f(b)$. For any $k \in [f(a), f(b)]$, there exists a point c in I such that $f(c) = k$.

5.3.10. (*Closed Interval Theorem*) If f is continuous on $[a, b]$, then $f([a, b]) \coloneqq \{f(x) :: x \in [a, b]\} = [m, M]$, where $m = \inf f([a, b])$ and $M = \sup f([a, b])$.

5.4. Uniform Continuity

5.4.1. (*Definition*) Let $A \subset \mathbb{R}$, $f: A \to \mathbb{R}$. We say that f is uniformly continuous on A if for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $\forall x, y \in A, |x - y| < \delta(\epsilon) \Rightarrow |f(x) - f(y)| < \epsilon$.

5.4.2. (*Sequential Criterion for Uniform Continuity*) The function $f: A \to \mathbb{R}$ is uniformly continuous on A if and only if for any two sequences (x_n), (y_n) in A such that $\lim_{n\to\infty} x_n - y_n = 0$, we have $\lim_{n\to\infty} f(x_n) - f(y_n) = 0$.

5.4.2. (*Nonuniform Continuity Criteria*) The following statements are equivalent:

1. f is not uniformly continuous on A.
2. $\exists \epsilon_0 > 0\ s.t. \forall \delta > 0, \exists x_\delta, y_\delta\ s.t. |x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$.
3. $\exists \epsilon_0 > 0, (x_n), (y_n)\ s.t. \lim_{n\to\infty} x_n - y_n = 0$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$.

5.4.3. (*Uniform Continuity Theorem*) If f is continuous on a closed bounded interval $[a, b]$, then it is uniformly continuous on $[a, b]$.

5.4.4. (*Lipschitz Condition*) A function $f: A \to \mathbb{R}$ is said to be Lipschitz function on A if there exists a $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|, \forall x, y \in A$.

5.4.5. (*Theorem*) If $f: A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.

5.4.7. (*Theorem of uniformly continuous functions preserve Cauchy sequence*) If $f: A \to \mathbb{R}$ is uniformly continuous on A and (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence.

5.4.8. (*Continuous Extension Theorem*) A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$.

5.6. Monotone and Inverse Functions

5.6.1. (*Definition*) The function $f: A \to \mathbb{R}$ is said to be increasing on A if whenever $x_1, x_2 \in A$, $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$. The function f is said to be strictly increasing if whenever $x_1, x_2 \in A$, $x_1 < x_2$, then $f(x_1) < f(x_2)$.

The function $f: A \to \mathbb{R}$ is said to be decreasing on A if whenever $x_1, x_2 \in A$, $x_1 \geq x_2$, then $f(x_1) \geq f(x_2)$. The function f is said to be strictly decreasing if whenever $x_1, x_2 \in A$, $x_1 > x_2$, then $f(x_1) > f(x_2)$.

If a function is either increasing or decreasing on A, we say that it is monotone on A. If f is either strictly increasing or strictly decreasing on A, we say that f is strictly monotone on A.

5.6.1. (*Theorem of one-sides limits for monotone functions exist*) Let $I \subset \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I. Then

(i) $\lim_{x \to c-} f(x) = \sup\{f(x) :: x \in I, x < c\}$.
(ii) $\lim_{x \to c+} f(x) = \inf\{f(x) :: x \in I, x > c\}$.

5.6.2. (*Corollary*) The following statements are equivalent:

a. f is continuous at c.
b. $\lim_{x \to c-} f(x) = f(c) = \lim_{x \to c+} f(x)$.
c. $\sup\{f(x) :: x \in I, x < c\} = f(c) = \inf\{f(x) :: x \in I, x > c\}$.

5.6.3. (*Definition*) If $f: I \to \mathbb{R}$ is increasing on I and if c is not an endpoint of I, we define the jump of f at c to be $j_f(c) \coloneqq \lim_{x \to c+} f(x) - \lim_{x \to c-} f(x) = \inf\{f(x) :: x \in I, x > c\} - \sup\{f(x) :: x \in I, x < c\}$.

5.6.3. (*Theorem*) Let $f: I \to \mathbb{R}$ be increasing on I. Then f is continuous at c if and only if $j_f(c) = 0$.

5.6.4. (*Theorem*) Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be monotone on I. Then the set of points $D \subseteq I$ at which f is discontinuous is a countable set.

5.6.5. (*Continuous Inverse Theorem*) Let $I \subset \mathbb{R}$ be an interval and $f: I \to \mathbb{R}$ be strictly monotone and continuous. Then the inverse function f^{-1} is also strictly monotone and continuous on J.

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