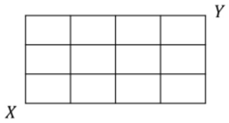
MA2214 Combinatorics and Graphs I

AY2021/22 Semester 1

Chapter 1 – Permutations and Combinations

- r-permutations of n distinct objects: $P_r^n = \frac{n!}{(n-r)!}$
- r-circular permutations of n distinct objects: $Q_r^n = \frac{n!}{(n-r)!} \cdot \frac{1}{r}$
- r-combinations of n distinct objects: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- r-permutations of n distinct objects with repetition allowed: n^r
- r-permutations of $M = \{r_1 \cdot a_1, r_2 \cdot a_2, \dots, r_n \cdot a_n\}$: $P(r: r_1, r_2, \dots, r_n) = \frac{r!}{r_1! r_2! \dots r_n!}$
- Number of r-element multi-subsets of $M = \{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_n\}$: $H_r^n = \binom{r+n-1}{r}$
  - Number of non-negative solutions of $x_1 + x_2 + \cdots + x_n = r$: $H_r^n = \binom{r+n-1}{r}$
- Identities:
  - $\binom{n}{r} = \binom{n}{n-r}$
  - $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$
  - $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$
  - $\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$
  - $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}$
- Number of shortest routes from $X(0,0)$ to $Y(m,n)$: $\binom{m+n}{n}$

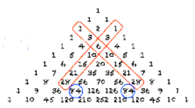


Chapter 2 – Binomial and Multinomial Coefficients

- (*Binomial Theorem*) $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$
  - $\sum_{r=0}^{n} \binom{n}{r} = 2^n$
  - $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$
  - $\sum_{r=1}^{n} r \binom{n}{r} = n \cdot 2^{n-1}$
  - $\sum_{r=1}^{n} r^2 \binom{n}{r} = n(n+1)2^{n-2}$
  - (*Vandermonde's Identity*) $\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}$
    - $\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}$
- Pascal's Triangle
  - (*Chu Shih-Chieh Identity*)

    (i) $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$

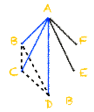
    (ii) $\binom{r}{0} + \binom{r+1}{1} + \cdots + \binom{r+k}{k} = \binom{r+k+1}{k}$



- (*Multinomial Theorem*) $(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots+n_m=n} \binom{n}{n_1, n_2, \dots, n_m} x_1^{n_1} x_2^{n_2} \dots x_m^{n_m}$

Chapter 3 – Pigeonhole Principle

- (*Pigeonhole Principle*) Let k and n be any two positive integers. If at least $kn+1$ objects are distributed among n boxes, then one of the boxes must contain at least $k+1$ objects. In particular, if at least $n+1$ objects are to be put into n boxes, then one of the boxes must contain at least two objects.
- (*Generalised Pigeonhole Principle*) Let $n, k_1, k_2, \dots, k_n \in \mathbb{N}$. If $k_1 + k_2 + \cdots + k_n - (n-1)$ or more objects are put into n boxes, then either
  - the first box contains at least k_1 objects; OR
  - the second box contains at least k_2 objects; OR
  - …
  - the n-th box contains at least k_n objects.
- (*Ramsey Number*) Let $R(p,q)$ denote the smallest natural number n such that for any colouring of the edges of an n-clique by 2 colours, blue and red, there exists either a "blue p-clique" or a "red q-clique".
  - $R(p,q) = R(q,p)$
  - $R(1,q) = 1$
  - $R(2,q) = q$
  - $R(3,3) = 6$
  - (*Theorem 23.1*) $R(p,q) \le R(p-1,q) + R(p,q-1)$
  - (*Theorem 23.2*) If $R(p-1,q)$ and $R(p,q-1)$ are even, then $R(p,q) \le R(p-1,q) + R(p,q-1) - 1$



p \ q	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40-43
4		18	25	35-41	49-61	56-84	73-115	92-149
5			43-49	58-87	80-143	101-216	126-316	144-442
6				102-165	113-298	132-495	169-780	179-1171
7					205-540	217-1031	241-1713	289-2826
8						282-1870	317-3583	?-6090
9							565-6588	581-12677
10								798-23556

Chapter 4 – Principle of Inclusion and Exclusion

- (*Principle of Inclusion and Exclusion*) $|A_1 \cup A_2 \cup \dots \cup A_q| = \sum_{i=1}^{q} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{q+1} |A_1 \cap A_2 \cap \dots \cap A_q|$
- (*Generalised Principle of Inclusion and Exclusion*) Number of elements of S that possesses exactly m of the q properties: $E(m) = \sum_{k=m}^{q} (-1)^{k-m} \binom{k}{m} \omega(k)$, where $\omega(k) = \sum \omega(p_{i1} p_{i2} \dots p_{ik})$
  - Number of elements without any properties: $E(0) = \omega(0) - \omega(1) + \cdots + (-1)^q \omega(q)$
- (*Stirling Number*) Number of ways to distribute r distinct objects into n identical boxes so that no box is empty: $S(r,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^r$
  - $S(0,0) = 1$
  - $S(r,0) = S(0,n) = 0$
  - $S(r,n) > 0$ if $r \ge n \ge 1$
  - $S(r,n) = 0$ if $n > r \ge 1$
  - $S(r,1) = 1$
  - $S(r,r) = 1$
  - $S(r,r-1) = \binom{r}{2}$
  - $S(r,r-2) = \binom{r}{3} + 3\binom{r}{4}$
  - $S(r,n) = S(r-1,n-1) + nS(r-1,n)$
  - Number of partitions of $\{1,2,\dots,r\}$: $\sum_{n=1}^{r} S(r,n)$
  - (*Theorem 27.1*) Number of surjective mappings from $\mathbb{N}_r$ to $\mathbb{N}_n$: $F(r,n) = n!\,S(r,n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^r$
- (*Theorem 28.2*) Number of r-permutations of n distinct objects with k fixed points: $D(r,n,k) = \frac{\binom{r}{k}}{(n-r)!} \sum_{i=0}^{r-k} (-1)^i \binom{r-k}{i} (n-k-i)!$
  - Number of derangements of $\mathbb{N}_n$: $D_n = D(n,n,0) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$
- Number of integers between 1 and n which are coprime to n: $\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right)$

Distribution Problems	
r distinct objects	**n distinct boxes**
• Each box can hold at most one object: P_r^n • Each box can hold any number of objects: n^r • Each box can hold any number of objects and the orderings of objects inside each box count: $n(n+1)\dots(n+(r-1))$	
r identical objects	**n distinct boxes**
• Each box can hold at most one object: $\binom{n}{r}$ • Each box can hold any number of objects: H_r^n • Each box holds at least one object: $\binom{r-1}{r-n}$	
r distinct objects	**n identical boxes**
• Each box holds at least one object: $S(r,n)$	
r identical objects	**n identical boxes**

Chapter 5 – Generating Functions

- Generalised binomial expansion: $(1 \pm x)^\alpha = \sum_{r=0}^{\infty} \binom{\alpha}{r} (\pm x)^r$
  - $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$
  - $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$
  - $(1-x)^{-n} = 1 + \binom{1+n-1}{1} x + \binom{2+n-1}{2} x^2 + \cdots + \binom{r+n-1}{r} x^r + \cdots$
- Ordinary generating function of $(a_r) = \{a_0, a_1, \ldots, a_r, \ldots\}$: $A(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + \cdots + a_r x^r + \cdots$
  - Generating function of $(a_r) = \{0, 0, \ldots, 0, 1, 0, \ldots\}$, where $a_n = 1$: $A(x) = x^n$
  - Generating function of $(a_r) = \left\{\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, 0, 0, \ldots\right\}$: $A(x) = (1+x)^n$
  - Generating function of $(a_r) = \{1, 1, \ldots\}$: $A(x) = \frac{1}{1-x}$
  - Generating function of $(a_r) = \{1, k, k^2, \ldots\}$: $A(x) = \frac{1}{1-kx}$
  - Generating function of $(a_r) = \{1, 2, 3, \ldots\}$: $A(x) = \frac{1}{(1-x)^2}$
  - Generating function of $(a_r) = \{\binom{n-1}{0}, \binom{1+n-1}{1}, \binom{2+n-1}{2}, \ldots\}$: $A(x) = (1-x)^{-n}$
- (*Theorem 31.1*) Power series operations: $A(x) + B(x) = c_0 + c_1 x + c_2 x^2 + \cdots$, where $c_r = a_r + b_r$ $A(x)B(x) = d_0 + d_1 x + d_2 x^2 + \cdots$, where $d_r = a_0 b_r + a_1 b_{r-1} + \cdots + a_r b_0$
  - Generating function of (c_r), where $c_r = \alpha a_r + \beta b_r$: $C(x) = \alpha A(x) + \beta B(x)$
  - Generating function of (c_r), where $c_r = a_0 b_r + a_1 b_{r-1} + \cdots + a_r b_0$: $C(x) = A(x)B(x)$
  - Generating function of (c_r), where $c_r = a_0 a_r + a_1 a_{r-1} + \cdots + a_r a_0$: $C(x) = A^2(x)$
  - Generating function of $(c_r) = \{0, 0, \ldots 0, a_0, a_1, \ldots\}$, where there are m 0s: $C(x) = x^m A(x)$
  - Generating function of (c_r), where $c_r = k^r a_r$: $C(x) = A(kx)$
  - Generating function of $(c_r) = \{a_0, a_1 - a_0, a_2 - a_1, \ldots\}$: $C(x) = (1-x)A(x)$
  - Generating function of $(c_r) = \{a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots\}$: $C(x) = \frac{A(x)}{1-x}$
  - Generating function of $(c_r) = \{a_1, 2a_2, 3a_3, \ldots\}$: $C(x) = A'(x)$
  - Generating function of $(c_r) = \{0, a_1, 2a_2, \ldots\}$: $C(x) = xA'(x)$
  - Generating function of $(c_r) = \{0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \ldots\}$: $C(x) = \int_0^x A(t)dt$
- OGF of r-combinations from the multiset $M = \{n_1 \cdot b_1, n_2 \cdot b_2, \ldots, n_k \cdot b_k\}$: $\prod_{i=1}^{k} (\sum_{j=0}^{n_i} x^j)$
  - Number of partitions of r into parts of size 1, 2 or 3: $(1 + x + x^2 + \cdots)(1 + x^2 + (x^2)^2 + \cdots)(1 + x^3 + (x^3)^2 + \cdots)$
  - Number of partitions of r into distinct parts of arbitrary size: $$(1+x)(1+x^2)(1+x^3)\ldots$$ Note that $1 + x = \frac{1-x^2}{1-x}$.
  - Number of partitions of r into odd parts: $$(1 + x + x^2 + \cdots)(1 + x^3 + (x^3)^2 + \cdots)\ldots$$
  - (*Euler*) Number of partitions of r into distinct parts is equal to number of partitions of r into odd parts.
- Exponential generating function of $(a_r) = \{a_0, a_1, \ldots, a_r, \ldots\}$: $A(x) = \sum_{r=0}^{\infty} a_r \frac{x^r}{r!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \cdots$
  - Generating function of $(a_r) = \{1, 1, \ldots\}$: e^x
  - Generating function of $(a_r) = \{0!, 1!, 2!, \ldots\}$: $\frac{1}{1-x}$
  - Generating function of $(a_r) = \{0, k, k^2, \ldots\}$: e^{kx}
  - Generating function of $(a_r) = \{P_0^n, P_1^n, P_2^n, \ldots\}$: $(1+x)^n$
- EGF of r-permutations from the multiset $M = \{n_1 \cdot b_1, n_2 \cdot b_2, \ldots, n_k \cdot b_k\}$: $\prod_{i=1}^{k} (\sum_{j=0}^{n_i} \frac{x^j}{j!})$
  - EGF of r-permutations of p blue identical balls and q red identical balls: $(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^p}{p!})(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^q}{q!})$
- Exponential operations
  - $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$
  - $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^r x^r}{r!} + \cdots$
  - $\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$
  - $\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

Chapter 6 – Recurrence Relations

- Solution to linear homogenous recurrence relations: If $\alpha 1, \alpha 2, \ldots, \alpha_k$ are the distinct characteristic roots such that α_i is of multiplicity m_i, then the general solution is given by $$a_n = \sum_{i=1}^{k} (A_{11} + A_{12} n + \cdots + A_{1,m_i} n^{m_i - 1})(\alpha_i)^n$$
- Complex roots
  - $\alpha = a + ib = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left|\frac{b}{a}\right|$.
  - (De Moivre's Theorem) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Chapter 7 – Basic Concepts of Graphs

- Order: Number of vertices
- Size: Number of edges
- Neighbourhood ($N(v_i)$): Set of all neighbours of v_i
- Closed neighbourbood ($N[v_i]$): Set of all neighbours union v_i
- (*Handshaking Lemma*) Sum of order of every vertex in any graph is even.
- Maximum size of a graph of order n: $\binom{n}{2}$
- Sum of degree of vertices: $2e$
- Subgraph: H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
- Spanning subgraph: H is a subgraph of G and $V(H) = V(G)$.
- Induced subgraph: $E(H) = \{uv \in E(G) \mid u \in V(H), v \in V(H)\}$
- Subgraph via deletion: When a set of edge is deleted, the subgraph is always a spanning graph; when a set of vertices is deleted, the subgraph is always an induced graph.
- (*Theorem*) Let G be a graph.
  - If $A \subseteq V(G)$, then $G - A = [V(G) - A]$.
  - Let H be a subgraph of G. H is an induced subgraph of G if and only if $H = G - (V(G) - V(H))$.
- Walk: Vertices and edges need not be distinct.
  - A walk is open if source is distinct from destination, otherwise closed.
- Trail: Vertices need not be distinct while edges are distinct.
- Path (P_n): Vertices are distinct, hence edges are distinct.
- Circuit: A closed trail of length at least 2
- Cycle (C_n): A closed path of length at least 2
- (*Theorem*) If a graph contains a u-v walk of length k, then it contains a u-v path of length at most k.
- Complement: $V(G) = V(\bar{G})$ and $\forall u, v, uv \in E(G) \Leftrightarrow uv \notin E(\bar{G})$
- (*Theorem*) If G is disconnected, then $\bar{G}$ is connected.
- Distance between u and v: Length of shortest u-v path
- Eccentricity: $e_G(u) = \max_{v \in V(G)} \{d(u,v)\}$
- Diameter: $\mathrm{diam}(G) = \max_{u \in V(G)} e(u) = \max_{u,v \in V(G)} \{d(u,v)\}$
- Radius: $\mathrm{rad}(G) = \min_{u \in V(G)} e(u)$, where such u is a central vertex. The centre of G is the subgraph of G induced by the set of central vertices.
- (*Triangle Inequality*) $d(u,v) \leq d(u,w) + d(w,v)$
- (*Theorem*) For any connected graph G, $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\mathrm{rad}(G)$
- v is a cut vertex in G if and only if $G - v$ is connected.
- Isomorphism: Two graphs are isomorphic if there exists a bijection $f: V(G) \to V(H)$ such that $uv \in E(G) \Leftrightarrow f(u)f(v) \in E(H)$.
- (*Theorem*) $G \cong H$ if and only if $\bar{G} \cong \bar{H}$.
- If $G \cong H$, then:
  - G and H must have the same order and size;
  - $\delta(G) = \delta(H)$ and $\Delta(G) = \Delta(H)$;
  - G and H must have the same degree sequence.

- (*Theorem*) Let $d = (d_1, d_2, \dots, d_n)$ be a non-increasing degree sequence. Denote $d^* = (d_2 - 1, d_3 - 1, \dots, d_{d_1+1} - 1, d_{d_1+2}, \dots, d_n)$. Then d is graphic if and only of d^* is graphic.
- Self-complementary: $G \cong \bar{G}$, the order $n = 4k$ or $4k + 1$.
- Adjacency matrix (A): row – set of vertices; column – set of vertices
  - A is symmetric.
  - Entries in A are either 0 or 1.
  - Sum of entries in the i-th row is $d(v_i)$.
  - The i-j entry of A^k is the number of different v_i-v_j walks of length k in G.
- Incidence matrix (M): row – set of vertices; column – set of edges
  - Each column contains exactly 2 1s.
  - Sum of entries in the i-th row is $d(v_i)$.

Chapter 8 – Bipartite Graphs and Trees

- Bipartite graph: A graph G is bipartite if its vertex set $V(G)$ can be partitioned into two disjoint subsets V_1 and V_2 such that each edge of G joins a vertex of V_1 to a vertex of V_2.
- Sum of degree of vertices in each partition is equal.
- Complete bipartite graph ($K_{p,q}$)
- Join: $V(G_1 + G_2) = V(G_1) \cup V(G_2)$; $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$
- (*Lemma*) Every closed walk of odd length p in a graph always contains an odd cycle.
- (*Theorem*) A graph is bipartite if and only if it contains no odd cycles.
- Tree: A connected graph is a tree if it contains no cycles.
  - Every two distinct vertices are joined by a unique path.
  - $\text{size} = \text{order} - 1$
- Forest: Each component is a tree.
- (*Theorem*) Let T be a tree and $\Delta(T) = k$. For $i = 1,2, \dots, k$, let n_i be number of vertices in T of degree i. Then $n_1 = 2 + n_3 + 2n_4 + \cdots + (k-2)n_k$.
- Number of non-isomorphic trees: 1, 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, …
- Spanning tree: A spanning subgraph that is a tree.
- Finding spanning tree: DFS, BFS
- (*Theorem*) A graph is connected if and only if it contains a spanning tree.
- (*Corollary*) If a graph is connected, then $e(G) \geq n - 1$.
- $G \circ e$: The multigraph of order $n - 1$ obtained from G by deleting all edges joining u and v, and by identifying u and v.
- Number of spanning trees: $\tau(G) = \tau(G - e) + \tau(G \circ e)$
  - Cycles: $\tau(C_n) = n$
  - G_1 and G_2 connected via a cut vertex/bridge: $\tau(G) = \tau(G_1)\tau(G_2)$
  - Two cycles sharing one common edge: $(p + q - 2) + (p - 1)(q - 1)$
  - Duplicate one edge of C_n: $2n - 1$
  - Two cycles sharing one common edge which is duplicated: $(p + q - 2) + 2(p - 1)(q - 1)$
- (*Matrix Tree Theorem*) Let G be a multigraph with $V(G) = \{v_1, v_2, \dots, v_n\}$. Let A be the adjacency matrix of G and C be the $n \times n$ diagonal matrix defined by

$$c_{ij} = \begin{cases} d(v_i) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

  Then $\tau(G)$ is equal to the cofactor of any entry in $C - A$.
- (*Theorem*) For $n \geq 2$, $\tau(K_n) = n^{n-2}$.
- Finding minimum spanning tree: Kruskal's and Prim's
- Computing distance: Dijkstra's