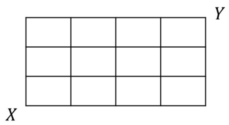
MA2214 Combinatorics and Graphs I

AY2021/22 Semester 1

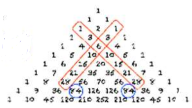
Chapter 1 – Permutations and Combinations

- r-permutations of n distinct objects: $P_r^n = \frac{n!}{(n-r)!}$
- r-circular permutations of n distinct objects: $Q_r^n = \frac{n!}{(n-r)!} \cdot \frac{1}{r}$
- r-combinations of n distinct objects: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- r-permutations of n distinct objects with repetition allowed: n^r
- r-permutations of $M = \{r_1 \cdot a_1, r_2 \cdot a_2, \ldots, r_n \cdot a_n\}$: $P(r: r_1, r_2, \ldots, r_n) = \frac{r!}{r_1! r_2! \ldots r_n!}$
- Number of r-element multi-subsets of $M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$: $H_r^n = \binom{r+n-1}{r}$
  - Number of non-negative solutions of $x_1 + x_2 + \cdots + x_n = r$: $H_r^n = \binom{r+n-1}{r}$
- Identities:
  - $\binom{n}{r} = \binom{n}{n-r}$
  - $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$
  - $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$
  - $\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$
  - $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}$
- Number of shortest routes from $X(0,0)$ to $Y(m,n)$: $\binom{m+n}{n}$



Chapter 2 – Binomial and Multinomial Coefficients

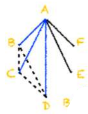
- (*Binomial Theorem*) $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$
  - $\sum_{r=0}^{n} \binom{n}{r} = 2^n$
  - $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$
  - $\sum_{r=1}^{n} r\binom{n}{r} = n \cdot 2^{n-1}$
  - $\sum_{r=1}^{n} r^2 \binom{n}{r} = n(n+1)2^{n-2}$
  - (*Vandermonde's Identity*) $\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}$
    - $\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}$
- Pascal's Triangle
  - (*Chu Shih-Chieh Identity*)
    (i) $\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$
    (ii) $\binom{r}{0} + \binom{r+1}{1} + \cdots + \binom{r+k}{k} = \binom{r+k+1}{k}$



- (*Multinomial Theorem*) $(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots+n_m=n} \binom{n}{n_1, n_2, \ldots, n_m} x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}$

Chapter 3 – Pigeonhole Principle

- (*Pigeonhole Principle*) Let k and n be any two positive integers. If at least $kn+1$ objects are distributed among n boxes, then one of the boxes must contain at least $k+1$ objects. In particular, if at least $n+1$ objects are to be put into n boxes, then one of the boxes must contain at least two objects.
- (*Generalised Pigeonhole Principle*) Let $n, k_1, k_2, \ldots, k_n \in \mathbb{N}$. If $k_1 + k_2 + \cdots + k_n - (n-1)$ or more objects are put into n boxes, then either
  - the first box contains at least k_1 objects; OR
  - the second box contains at least k_2 objects; OR
  - …
  - the n-th box contains at least k_n objects.
- (*Ramsey Number*) Let $R(p,q)$ denote the smallest natural number n such that for any colouring of the edges of an n-clique by 2 colours, blue and red, there exists either a "blue p-clique" or a "red q-clique".
  - $R(p,q) = R(q,p)$
  - $R(1,q) = 1$
  - $R(2,q) = q$
  - $R(3,3) = 6$
  - (*Theorem 23.1*) $R(p,q) \le R(p-1,q) + R(p,q-1)$
  - (*Theorem 23.2*) If $R(p-1,q)$ and $R(p,q-1)$ are even, then $R(p,q) \le R(p-1,q) + R(p,q-1) - 1$



p \ q	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40-43
4		18	25	35-41	49-61	56-84	73-115	92-149
5			43-49	58-87	80-143	101-216	126-316	144-442
6				102-165	113-298	132-495	169-780	179-1171
7					205-540	217-1031	241-1713	289-2826
8						282-1870	317-3583	?-6090
9							565-6588	581-12677
10								798-23556

Chapter 4 – Principle of Inclusion and Exclusion

- (*Principle of Inclusion and Exclusion*) $|A_1 \cup A_2 \cup \ldots \cup A_q| = \sum_{i=1}^{q} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{q+1} |A_1 \cap A_2 \cap \ldots \cap A_q|$
- (*Generalised Principle of Inclusion and Exclusion*) Number of elements of S that possesses exactly m of the q properties: $E(m) = \sum_{k=m}^{q} (-1)^{k-m} \binom{k}{m} \omega(k)$, where $\omega(k) = \sum \omega(p_{i1} p_{i2} \ldots p_{ik})$
  - Number of elements without any properties: $E(0) = \omega(0) - \omega(1) + \cdots + (-1)^q \omega(q)$
- (*Stirling Number*) Number of ways to distribute r distinct objects into n identical boxes so that no box is empty: $S(r,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^r$
  - $S(0,0) = 1$
  - $S(r,0) = S(0,n) = 0$
  - $S(r,n) > 0$ if $r \ge n \ge 1$
  - $S(r,n) = 0$ if $n > r \ge 1$
  - $S(r,1) = 1$
  - $S(r,r) = 1$
  - $S(r,r-1) = \binom{r}{2}$
  - $S(r,r-2) = \binom{r}{3} + 3\binom{r}{4}$
  - $S(r,n) = S(r-1,n-1) + nS(r-1,n)$
  - Number of partitions of $\{1,2,\ldots,r\}$: $\sum_{n=1}^{r} S(r,n)$
  - (*Theorem 27.1*) Number of surjective mappings from $\mathbb{N}_r$ to $\mathbb{N}_n$: $F(r,n) = n!\,S(r,n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^r$
- (*Theorem 28.2*) Number of r-permutations of n distinct objects with k fixed points: $D(r,n,k) = \frac{\binom{r}{k}}{(n-r)!} \sum_{i=0}^{r-k} (-1)^i \binom{r-k}{i} (n-k-i)!$
  - Number of derangements of $\mathbb{N}_n$: $D_n = D(n,n,0) = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$
- Number of integers between 1 and n which are coprime to n: $\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_k}\right)$

Distribution Problems	
r distinct objects	**n distinct boxes**
• Each box can hold at most one object: P_r^n • Each box can hold any number of objects: n^r • Each box can hold any number of objects and the orderings of objects inside each box count: $n(n+1)\ldots(n+(r-1))$	
r identical objects	**n distinct boxes**
• Each box can hold at most one object: $\binom{n}{r}$ • Each box can hold any number of objects: H_r^n • Each box holds at least one object: $\binom{r-1}{r-n}$	
r distinct objects	**n identical boxes**
• Each box holds at least one object: $S(r,n)$	
r identical objects	**n identical boxes**