MA3205 Set Theory

AY2022/23 Semester 1 · *Prepared by Tian Xiao @snoidetx*

Morse-Kelley Set Rules

1. Everything is a class.
2. Every set is a class; every class is a collection of sets; a class is a set if and only if it is a member of some class.
3. Every collection of sets is a class.
4. If A is a class and x is a set, then $A \cap x$ is a set.
5. The image of a set under a function is a set.
6. If A and B are sets, then so are $\{A, B\}$, $\cup A$ and $\mathcal{P}(A)$.
7. (*Axiom of Choice*) If $\langle A_i : i \in I \rangle$ is any sequence of sets such that $\forall i \in I\, [A_i \neq \emptyset]$, then $\prod_{i \in I} A_i \neq \emptyset$.
8. (*Axiom of Infinity*) $\mathbb{N}$ is a set.
9. (*Axiom of Extensibility*) $A = B \Leftrightarrow \forall x\, [x \in A \Leftrightarrow x \in B]$.

Basics

D1.11. $x \triangle y = x \backslash y \cup y \backslash x$.
T5.7. (*Cantor*) For any set X, $X \not\lessapprox \mathcal{P}(X)$.
D5.12. (*Schröder-Bernstein*) $A \lessapprox B \wedge B \lessapprox A \Rightarrow A \approx B$.
D6.4. $<$ is a partial order on X if
(1) $\forall x \in X\, [x \nless x]$;
(2) $\forall x, y, z \in X\, [(x < y \wedge y < z) \Rightarrow (x < z)]$.
D6.5. A partial order $\langle X, < \rangle$ is called a linear order if $\forall x, y \in X\, [(x = y) \vee (x < y) \vee (y < x)]$.
D6.13. A linear order $\langle X, < \rangle$ is called a well order if every non-empty subset of X has a minimal element.
D6.16. Let $\langle X, < \rangle$ be a linear order. For any $x \in X$ define $\mathrm{pred}_{\langle X, < \rangle}(x) = \{x' \in X : x' < x\}$.
D6.33. If $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ are linear orders, then a function $f : X \to Y$ is an *isomorphism* between $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ if the following hold:
(1) f is *1-1* and *onto*;
(2) $\forall x, y \in X\, [x \lhd y \Leftrightarrow f(x) \prec f(y)]$.
L6.34. $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ are linear orders. Suppose $f : X \to Y$ is an *onto* function such that $\forall x, y \in X\, [x \lhd y \Rightarrow f(x) \prec f(y)]$. Then f is an isomorphism.
D6.35. $\langle X, < \rangle$ and $\langle Y, \prec \rangle$ are linear orders. A function $f : X \to Y$ is called an *embedding* if $\forall x, x' \in X\, [x < x' \leftrightarrow f(x) \prec f(x')]$ and f is *1-1*. If there exists such embedding, $\langle X, < \rangle \hookrightarrow \langle Y, \prec \rangle$.
D6.42. A linear order $\langle X, \lhd \rangle$ has *type omega* if X is infinite and for every $x \in X$, $\mathrm{pred}_{\langle X, \lhd \rangle}(x)$ is finite.
T7.14. $2^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}} \approx \mathcal{P}(\mathbb{N}^{\mathbb{N}}) \approx \mathcal{P}(\mathbb{N}) \approx \mathcal{P}(\mathbb{Q}) \approx \mathbb{R}$.
D8.4. Suppose $\langle X, < \rangle$ is a finite partial order and $A \subseteq X$.

$$\begin{cases} \text{Upper bound } x\text{: } \forall a \in A\, [a \leq x]. \\ \text{Lower bound } x\text{: } \forall a \in A\, [x \leq a]. \\ \text{Supremum } u\text{: } \forall x \in \{\text{upper bounds}\}\, [u \leq x]. \\ \text{Infimum } u\text{: } \forall x \in \{\text{lower bounds}\}\, [x \leq u]. \end{cases}$$

Ordinals

D10.1. A set x is called *transitive* if $\forall y\, [y \in x \Rightarrow y \subseteq x]$.
D10.2. A set α is an *ordinal* if it is transitive and well-ordered by $\in$. Let $\in_\alpha$ denote $\{\langle \beta, \gamma \rangle \in \alpha \times \alpha : \beta \in \gamma\}$, then α is an ordinal if α is transitive and $\langle \alpha, \in_\alpha \rangle$ is a well order.
F10.3. $\mathbb{N}$ is an ordinal. Every $n \in \mathbb{N}$ is also an ordinal.
T10.4. Let x be an ordinal, then:

$$\begin{cases} \forall y \in x\, [y \text{ is an ordinal } \wedge y = pred_{\langle x, \in \rangle}(y)]; \\ y \text{ is any ordinal } \wedge \langle x, \in \rangle \text{ is isomorphic to } \langle y, \in \rangle \Rightarrow x = y; \\ y \text{ is any ordinal } \Rightarrow x \in y \vee x = y \vee y \in x; \\ y, z \text{ are any ordinals } \Longrightarrow x \in y \wedge y \in z \Rightarrow x \in z; \\ \exists y \in \mathbf{C} \exists z \in \mathbf{C}\, [y \in z \vee y = z], \text{ where } \mathbf{C} \text{ is a non-empty class of ordinals.} \end{cases}$$

D10.5. $\mathbf{ORD} = \{\alpha : \alpha \text{ is an ordinal}\}$.
T10.6. (*Burali-Forti*) **ORD** is not a set.
L10.7. Every transitive set if ordinals is an ordinal.
T10.8. Let $\langle X, < \rangle$ be a well-ordered set. Then there exists a unique ordinal α such that $\langle X, < \rangle$ is isomorphic to $\langle \alpha, \in_\alpha \rangle$.
D10.11. If $\langle X, < \rangle$ is any well-ordered set, then $\mathrm{otp}(X) = \mathrm{otp}(\langle X, < \rangle)$, which is called the *order type* of $\langle X, < \rangle$, is the unique ordinal α such that $\langle X, < \rangle$ is isomorphic to $\langle \alpha, \in_\alpha \rangle$.
L10.13. $\alpha \leq \beta \Leftrightarrow \alpha \subseteq \beta$.
L10.14. $\begin{cases} \text{If } A \text{ is a non-empty set of ordinals, then } \min(A) = \bigcap A; \\ \text{If } A \text{ is any set of ordinals, then } \sup_{\mathbf{ORD}}(A) = \bigcup A. \end{cases}$
L10.15. For any α, $\begin{cases} S(\alpha) \text{ is an ordinal}; \\ \alpha < S(\alpha); \\ \forall \beta\, [\beta < S(\alpha) \Leftrightarrow \beta \leq \alpha]. \end{cases}$
D10.16. $\begin{cases} \alpha \text{ is a } \textit{successor ordinal} \text{ if } \exists \beta\, [\alpha = S(\beta)]; \\ \alpha \text{ is a } \textit{limit ordinal} \text{ if } \alpha \neq 0 \wedge \alpha \text{ is not a successor ordinal.} \end{cases}$
L10.17. An ordinal α is a natural number if and only if $\forall \beta \leq \alpha\, [\beta = 0 \vee \beta \text{ is a successor ordinal}]$.
Conv. ω denotes the set of natural numbers ($\omega = \mathbb{N}$).
E10.27. $X \subseteq \alpha \Rightarrow \mathrm{otp}(\langle X, \in \rangle) \leq \alpha$.
E10.28. $\alpha > 0$ is a limit ordinal if and only if $\bigcup \alpha = \alpha$.

Induction and Recursion

T10.19. Let $P(\alpha)$ be some property. If $\forall \alpha \in \mathbf{ORD}\, [\forall \beta < \alpha\, [P(\beta)] \Rightarrow P(\alpha)]$, then $\forall \alpha \in \mathbf{ORD}\, [P(\alpha)]$.
D10.20. Let **FOD** denote the class of all functions whose domain is some ordinal, i.e.

$$\mathbf{FOD} = \{\sigma : \sigma \text{ is a function } \wedge \exists \alpha \in \mathbf{ORD}\, [\mathrm{dom}(\sigma) = \alpha]\}.$$

An *ordinal extender* is a function $\mathbf{E} : \mathbf{FOD} \to \mathbf{V}$.
T10.21. $\exists! \mathbf{F} : \mathbf{ORD} \to \mathbf{V}\, [\forall \alpha \in \mathbf{ORD}\, [\mathbf{F}(\alpha) = \mathbf{E}(\mathbf{F} \restriction \alpha)]]$.
E10.26. A class $\mathbf{C}$ is *trans-finitely inductive* if (1) $0 \in \mathbf{C}$ (2) $\forall x \in \mathbf{C}\, [S(x) \in \mathbf{C}]$ (3) $\forall X \subseteq \mathbf{C}\, [\bigcup X \in C]$. Then **ORD** is the smallest trans-finitely inductive class.

Ordinal Addition

D11.1. Let $\langle X, <_X \rangle$ and $\langle Y, <_Y \rangle$ be well orders. Define $X \oplus Y$ to be the set $(\{0\} \times X) \cup (\{1\} \times Y)$. Define $<_{X \oplus Y}$ by the following clauses:

$$\begin{cases} \forall x, x' \in X\, [\langle 0, x \rangle <_{X \oplus Y} \langle 0, x' \rangle \Leftrightarrow x <_X x']; \\ \forall y, y' \in Y\, [\langle 1, y \rangle <_{X \oplus Y} \langle 1, y' \rangle \Leftrightarrow y <_Y y']; \end{cases}$$

D11.2. $\alpha + \beta = \mathrm{otp}(\langle \alpha \oplus \beta, <_{\alpha \oplus \beta} \rangle)$.
L11.4. Let $\langle X, <_X \rangle$, $\langle Y, <_Y \rangle$, $\langle Z, <_Z \rangle$ be well orders. Suppose that $A, B \subseteq Z$. Assume that $A \cup B = Z$ and $\forall a \in A\, \forall b \in B\, [a <_Z b]$. Then if $\langle A, <_Z \rangle$ is isomorphic to $\langle X, <_X \rangle$ and $\langle B, <_Z \rangle$ is isomorphic to $\langle Y, <_Y \rangle$, then $\langle Z, <_Z \rangle$ is isomorphic to $\langle X \oplus Y, <_{X \oplus Y} \rangle$.
L11.5. $\begin{cases} \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma; \\ \alpha + 0 = \alpha; \\ \alpha + 1 = S(\alpha); \\ \alpha + S(\beta) = S(\alpha + \beta); \\ \beta \text{ is a limit ordinal} \Rightarrow \alpha + \beta = \sup\{\alpha + \xi : \xi < \beta\}. \end{cases}$
E11.13. $\alpha < \beta \Rightarrow (\gamma + \alpha < \gamma + \beta) \wedge (\alpha + \gamma \leq \beta + \gamma)$.
E11.14. If $\alpha \geq \omega$, then $1 + \alpha = \alpha$.

Ordinal Multiplication

D11.7. $\alpha \cdot \beta = \mathrm{otp}(\langle \beta \times \alpha, <_{\alpha \cdot \beta} \rangle)$. $<_{\alpha \cdot \beta}$ is dictionary order.
L11.8. Suppose $A \subseteq \gamma$ and $\langle A, \in \rangle$ is isomorphic to $\langle \beta, \in \rangle$. Then $\langle A \times \alpha, <_{\alpha \times \gamma} \rangle$ is isomorphic to $\langle \beta \times \alpha, <_{\alpha \times \beta} \rangle$.
L11.9. $\begin{cases} \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma; \\ \alpha \cdot 0 = 0; \\ \alpha \cdot 1 = \alpha; \\ \beta \text{ is a limit ordinal} \Rightarrow \alpha \cdot \beta = \sup\{\alpha \cdot \xi : \xi < \beta\}; \\ \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma. \end{cases}$
E11.12. $\forall \alpha > 0\, [\alpha \cdot \omega > \alpha]$.
E11.15. If $\gamma > 0$, then $\alpha < \beta \Rightarrow (\gamma \cdot \alpha < \gamma \cdot \beta) \wedge (\alpha \cdot \gamma \leq \beta \cdot \gamma)$.
E11.16. $0 < \alpha \leq \beta \longrightarrow \exists! \delta, \xi\, [\xi < \alpha \wedge \alpha \cdot \delta + \xi = \beta]$.

Ordinal Exponentiation

D11.10. For a fixed α, define α^β recursively on β using the following clauses:

$$\begin{cases} \alpha = 0 \Rightarrow^0 = 0; \alpha > 0 \Rightarrow \alpha^0 = 1; \\ \alpha^{\beta + 1} = \alpha^\beta \cdot \alpha; \\ \beta \text{ is a limit ordinal} \Rightarrow \alpha^\beta = \sup\{\alpha^\xi : \xi < \beta\}. \end{cases}$$

Cardinals

D12.1. A set X is said to be *well-orderable* if there exists a relation $< \subseteq X \times X$ such that $\langle X, < \rangle$ is a well order.
D12.2. The *cardinality* of X is $|X| = \min\{\alpha \in \mathbf{ORD} : \alpha \approx X\}$.
D12.3. α is a *cardinal* if $|\alpha| = \alpha$.
F12.4. ω is a cardinal. If $n \in \omega$, n is a cardinal.
L12.5. If $|\alpha| < \beta < \alpha$. then $|\beta| = |\alpha|$.
L12.6. X is finite $\Leftrightarrow |X| < \omega$; X is countable $\Leftrightarrow |X| \leq \omega$.

D12.7. Let κ and λ be cardinals. Both $(\{0\}\times\kappa)\cup(\{1\}\times\lambda)$ and $\kappa\times\lambda$ are well-orderable. Define $\begin{cases}\kappa\boxplus\lambda=|(\{0\}\times\kappa)\cup(\{1\}\times\lambda)|;\\ \kappa\boxtimes\lambda=|\kappa\times\lambda|.\end{cases}$

L12.8. Every infinite cardinal is a limit cardinal.

T12.9. If κ is an infinite cardinal, then $\kappa\boxtimes\kappa=\kappa$.

C12.10. Let κ and λ be infinite cardinals. $\kappa\boxplus\lambda=\kappa\boxtimes\lambda=\max\{\kappa,\lambda\}$.

T12.11. For every set X there is a cardinal α such that there is no *1-1* function $f:\alpha\to X$.

L12.15. Let A be any set of cardinals. Then $\bigcup A$ is a cardinal.

D12.16. For each $\alpha\in\mathbf{ORD}$, α^+ is the least cardinal strictly greater than α.

L12.17. Suppose $\mathbf{F}:\mathbf{ORD}\to\mathbf{ORD}$ is a function such that $\forall\alpha,\beta\in\mathbf{ORD}\,[\alpha<\beta\Rightarrow\mathbf{F}(\alpha)<\mathbf{F}(\beta)]$. Then $\forall\beta\in\mathbf{ORD}\,[\beta\le\mathbf{F}(\beta)]$.

D12.18. Define a sequence $\langle\omega_\alpha:\alpha\in\mathbf{ORD}\rangle$ by induction using the following clauses:

$$\begin{cases}\omega_0=\omega;\\ \omega_{S(\alpha)}=\omega_\alpha^+;\\ \alpha\text{ is a limit ordinal}\Rightarrow\omega_\alpha=\sup\{\omega_\xi:\xi<\alpha\}.\end{cases}$$

ω_α is also denoted as $\aleph_\alpha$.

D12.20. $\begin{cases}\alpha<\beta\Rightarrow\aleph_\alpha<\aleph_\beta.\\ \text{Every infinite cardinal is equal to }\aleph_\alpha\text{, for some }\alpha\in\mathbf{ORD}.\end{cases}$

Choice and Cardinality

D12.21. Let X be any set. We say that F is a *choice function* on X if F is a function, $\mathrm{dom}(F)=X\setminus\{0\}$ and $\forall a\in X\setminus\{0\}\,[F(a)\in a]$.

T12.22. (*Zermelo*) X is well-orderable $\Leftrightarrow$ there exists a choice function on $\mathcal{P}(X)$.

T12.26. The following statements are equivalent:

$$\begin{cases}\text{The Cartesian product of non-empty sets is non-empty;}\\ \text{For every set } X \text{ there exists a choice function on } X;\\ \text{Every set is well-orderable;}\\ \text{For any two sets } X,Y,\text{ either } X\lesssim Y \text{ or } Y\lesssim X;\\ \text{For any set } X \text{ there is any ordinal } \alpha \text{ and a } \textit{1-1}\ f:X\to\alpha;\\ \text{For any set } X \text{ there is a cardinal } \kappa \text{ such that } X\approx\kappa.\end{cases}$$

Cardinal Exponentiation (AC)

D12.28. $\kappa^\lambda=|\{f: f\text{ is a function }\wedge\mathrm{dom}(f)=\lambda\wedge\mathrm{ran}(f)\subseteq\kappa\}|$.

L12.30. $\begin{cases}(\kappa^\lambda)^\theta=\kappa^{\lambda\boxtimes\theta};\\ (\kappa^\lambda)\boxtimes(\kappa^\theta)=\kappa^{\lambda\boxplus\theta}.\end{cases}$

D12.31. Define a sequence $\langle\beth_\alpha:\alpha\in\mathbf{ORD}\rangle$ by induction using the following clauses:

$$\begin{cases}\beth_0=\omega;\\ \beth_{S(\alpha)}=2^{\beth_\alpha};\\ \alpha\text{ is a limit ordinal}\Rightarrow\beth_\alpha=\sup\{\beth_\xi:\xi<\alpha\}.\end{cases}$$

D12.32. $\begin{cases}(\textit{Generalised Continuum Hypothesis})\ \forall\alpha\in\mathbf{ORD}\,[\beth_\alpha=\aleph_\alpha].\\ (\textit{Continuum Hypothesis})\ \beth_1(=2^{\beth_0}=2^{\aleph_0})=\aleph_1.\end{cases}$

T12.34. (*König*) ${\aleph_\omega}^{\aleph_0}>\aleph_\omega$.

C12.35. $2^{\aleph_0}\ne\aleph_\omega$.

Applications of AC

D13.1. Let A be any set. $\mathcal{F}\subseteq\mathcal{P}(A)$ is of *finite character* if and only if $\forall X\subseteq A\,[X\in\mathcal{F}\iff\forall Y\subseteq X\,[|Y|<\omega\Rightarrow Y\in\mathcal{F}]]$, i.e. $X\in\mathcal{F}$ if and only if all its finite subsets are in $\mathcal{F}$.

L13.2. If $\mathcal{F}\subseteq\mathcal{P}(A)$ is of finite character, then $X\in\mathcal{F}\wedge Y\subseteq X\Rightarrow Y\in\mathcal{F}$.

T13.3. The following statements are equivalent:

- **AC.**
- (*Teichmüller-Tukey Lemma*) **For any set A and $\mathcal{F}\subseteq\mathcal{P}(A)$, if $\mathcal{F}$ has finite character, then for every $X\in\mathcal{F}$, there exists $Y\subseteq\mathcal{F}$ such that $X\subseteq Y$ and Y is maximal in $\langle\mathcal{F},\subsetneq\rangle$.** By AC fix an ordinal α and an *1-1* and *onto* function $e:\alpha\to A$. Define a function $f:\alpha\to 2$ by induction on α: fix $\xi<\alpha$ and suppose $f(\xi)$ has been defined for $\zeta<\xi$. If $X\cup\{e(\zeta):\zeta<\xi\wedge f(\zeta)=1\}\cup\{e(\xi)\}\in\mathcal{F}$, then define $f(\xi)=1$; otherwise $f(\xi)=0$. Let $Y=\{e(\xi):\xi<\alpha\wedge f(\xi)=1\}$. First check $X\cup Y\in\mathcal{F}$. Next check $X\subseteq Y$. Finally, there is no $Z\in\mathcal{F}$ such that $Y\subsetneq Z$. So Y is as required.
- (*Hausdorff's Maximal Chain Theorem*) **Every chain in every partial order is contained in a maximal chain.** Suppose $\langle X,<\rangle$ is a partial order and $C\subseteq X$ is a chain. $\mathcal{F}=\{A\subseteq X: A\text{ is a chain}\}$ has finite character. By Teichmüller-Tukey Lemma $\exists A\in\mathcal{F}\,[C\subseteq A\wedge A\text{ is maximal in }\langle\mathcal{F},\subsetneq\rangle]$. A is a maximal chain containing C.
- (*Zorn's Lemma*) **If $\langle X,<\rangle$ is any partial order which has the property that every chain in $\langle X,<\rangle$ has an upper bound in $\langle X,<\rangle$, then $\langle X,<\rangle$ has a maximal element.** Suppose $\langle X,<\rangle$ is a partial order such that every chain has an upper bound. $\emptyset$ is a chain. By Hausdorff's Maximal Chain Theorem, $\exists C\subseteq X\,[C\text{ is a maximal chain}]$. C has an upper bound $x\in X$. x is maximal in $\langle X,<\rangle$. If not, then $\exists y\ne x\,[x<y]$. $C\cup\{y\}$ is a chain, contradicting the maximality of C.

Appendix

Extender. Generally, suppose we have a function $\mathbf{F}:\mathbf{ORD}\to V$ that is defined recursively as (1) $\mathbf{F}(0)=v_0$ (2) given $\mathbf{F}(\alpha)\in\mathbf{ORD}$, $\mathbf{F}(\alpha+1)=h(\mathbf{F}(\alpha))$ (3) if α is a limit ordinal, then $\mathbf{F}(\alpha)=\sup\{\mathbf{F}(\xi):\xi<\alpha\}$, then the extender corresponding to $\mathbf{F}$ should be defined as

$$\mathbf{E}(\sigma)=\begin{cases}v_0 & \mathrm{dom}(\sigma)=0\\ h(\sigma(\beta)) & \mathrm{dom}(\sigma)=S(\beta),\exists\beta\in\mathbf{ORD}\wedge\sigma(\beta)\in\mathbf{ORD}\\ 0 & \mathrm{dom}(\sigma)=S(\beta),\exists\beta\in\mathbf{ORD}\wedge\sigma(\beta)\notin\mathbf{ORD}\\ \bigcup\mathrm{ran}(\sigma) & \mathrm{dom}(\sigma)\text{ is a limit ordinal}\end{cases}.$$

$\lambda^\lambda=2^\lambda$. Take $f\in\lambda^\lambda$, then $f\subseteq\lambda\times\lambda$. Since $\lambda\boxtimes\lambda=\lambda$, there exists a bijection $e:\lambda\times\lambda\to\lambda$. $\mathrm{Im}_e(f)$ provides a bijection between λ^λ and 2^λ.

$\langle P,\subsetneq\rangle$ satisfies Zorn's Lemma. First, verify that $\langle P,\subsetneq\rangle$ is a partial order by **D6.4**. Next, let $C\subseteq P$ be a chain. Let $\mathcal{F}=\bigcup C$ be an upper bound of C and show $\mathcal{F}\in P$ by contradiction. By Zorn's Lemma, every chain in P has a maximal element.

E11.17. $\alpha>0$ is an ordinal. Then $\alpha^{\beta+\gamma}=\alpha^\beta\cdot\alpha^\gamma$.

E11.x. Let X be any set of ordinals, α be any ordinal.

$$\begin{cases}(X\ne\emptyset)\wedge(\forall\alpha\in X\,[\alpha\text{ limit ordinal}])\Rightarrow\sup(X)\text{ limit ordinal;}\\ \sup(X)\text{ successor ordinal}\Rightarrow\sup(X)\in X;\\ \alpha\cdot\omega\le\omega^\alpha.\end{cases}$$

E12.36. Let κ,λ be infinite cardinals with $\lambda\le\kappa$. Then $\kappa^\lambda=|\{X\subseteq\kappa:|X|=\lambda\}|$.

E12.37. Let $\kappa,\lambda,\theta,\chi$ be cardinals. If $\kappa\le\lambda$, then $\kappa^\theta\le\lambda^\theta$; if $\kappa\le\chi,\lambda\le\theta$ and $\lambda\ne0$, then $\kappa^\lambda\le\chi^\theta$.

E12.39. $\begin{cases}\text{There exists a cardinal }\kappa\text{ such that }\aleph_\kappa=\kappa.\\ \text{There exists a cardinal }\kappa\text{ such that }\beth_\kappa=\kappa.\end{cases}$

E12.x Let κ,λ be infinite cardinals.

$$\begin{cases}\kappa\le\lambda\Rightarrow\kappa^\lambda=2^\lambda;\\ (\aleph_1)^{\aleph_0}=2^{\aleph_0}.\end{cases}$$

Legends

C	Corollary
D	Definition
E	Exercise
F	Fact
L	Lemma
T	Theorem
Conv.	Convention