MA3205 Set Theory

AY2022/23 Semester 1 · *Prepared by Tian Xiao @snoidetx*

Morse-Kelley Set Rules

1. Everything is a class.
2. Every set is a class; every class is a collection of sets; a class is a set if and only if it is a member of some class.
3. Every collection of sets is a class.
4. If A is a class and x is a set, then $A \cap x$ is a set.
5. The image of a set under a function is a set.
6. If A and B are sets, then so are A, B, $\bigcup A$ and $\mathcal{P}(A)$.
7. (*Axiom of Choice*) If $\langle A_i : i \in I\rangle$ is any sequence of sets such that $\forall i \in I\,[A_i \neq \emptyset]$, then $\prod_{i\in I} A_i \neq \emptyset$.
8. (*Axiom of Infinity*) $\mathbb{N}$ is a set.
9. (*Axiom of Extensibility*) $A = B \Leftrightarrow \forall x\,[x \in A \Leftrightarrow x \in B]$.

Set Operations

Subset $\subseteq$

D1.6. $A \subseteq B$ if $\forall x\,[x \in A \Rightarrow x \in B]$.

Empty Set $\emptyset$

D1.7. A set x is empty if $\forall y\,[y \notin x]$.
F1.8. If $x = \emptyset$ and A is any collection, then $x \subseteq A$.
F1.9. If x and y are empty sets, then $x = y$.

Union $\cup$ and Intersection $\cap$

D1.11. $\begin{cases} x \cup y = \{z : z \in x \vee z \in y\} \\ x \cap y = \{z : z \in x \wedge z \in y\} \end{cases}$

D1.13. $\begin{cases} \bigcup A & = \{x : \exists y\,[y \in A \wedge x \in y]\} \\ \bigcap A & = \begin{cases} 0 & \text{if } A = \emptyset; \\ \{x : \forall y\,[y \in A \Rightarrow x \in y]\} & \text{otherwise.} \end{cases} \end{cases}$

Other Operators $\backslash, \triangle, \mathcal{P}$

D1.11. $\begin{cases} x \backslash y = \{z : z \in x \wedge z \notin y \\ x \triangle y = x \backslash y \cup y \backslash x \\ \mathcal{P}(x) = \{z : z \subseteq x\} \end{cases}$

Commutativity	$x \cup y = y \cup x$ $x \cap y = y \cap x$
Associativity	$x \cup (y \cup z) = (x \cup y) \cup z$ $x \cap (y \cap z) = (x \cap y) \cap z$
Distributivity	$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$ $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$
De Morgan	$x \backslash (y \cup z) = (x \backslash y) \cap (x \backslash z)$ $x \backslash (y \cap z) = (x \backslash y) \cup (x \backslash z)$

E1.16. $\begin{cases} x \triangle \emptyset = x; x \triangle x = \emptyset \\ x \triangle y = y \triangle x \\ (x \triangle y) \triangle z = x \triangle (y \triangle z) \end{cases}$
E1.18. $x \cap (y \triangle z) = (x \cap y) \triangle (x \cap z)$.

Relations and Functions

Ordered Pair $\langle a, b\rangle$

D2.1. An ordered pair $\langle a, b\rangle$ is the set $\{\{a\}, \{a, b\}\}$.
L2.2. $\langle x, y\rangle = \langle a, b\rangle \Leftrightarrow (x = a) \wedge (y = b)$.
D2.3. $A \times B = \{z : \exists a \in A\, \exists b \in B\,[z = \langle a, b\rangle]\}$.

Relation R

D2.6. A relation R is a collection of ordered pairs ($\forall x \in R\, \exists a\, \exists b\,[x = \langle a, b\rangle]$).

- R is a relation on A if $R \subseteq A \times A$.
- $\operatorname{dom}(R) = \{a : \exists b\,[\langle a, b\rangle \in R]\}$.
- $\operatorname{ran}(R) = \{b : \exists a\,[\langle a, b\rangle \in R]\}$.
- $R^{-1} = \{x : \exists a\, \exists b\,[\langle a, b\rangle \in R \wedge x = \langle b, a\rangle]\}$.

F2.9. If R is a relation and $S \subseteq R$, then S is a relation.
D2.10. If R is a relation and A is any collection, then R restricted to A, $R \restriction A$, is $R \cap (A \times \operatorname{ran} R)$.
D2.12. $\operatorname{Im}_R(A) = \{b : \exists a \in A\,[\langle a, b\rangle \in R]\}$.
L2.15. Let R be a relation and A be a collection, then $\operatorname{Im}_R(\bigcup A) = \bigcup(I : \exists a \in A\,[I = \operatorname{Im}_R(a)])$.
L2.16. Let R be a relation such that $\forall x, z\,[x \neq z \Rightarrow \operatorname{Im}_R(\{x\}) \cap \operatorname{Im}_R(\{y\}) = \emptyset]$. Let A and B be any collections, then:

- $\operatorname{Im}_R(\bigcap A) = \bigcap\{I : \exists a \in A\,[I = \operatorname{Im}_R(a)]\}$.
- $\operatorname{Im}_R(B \backslash A) = \operatorname{Im}_R(B) \backslash \operatorname{Im}_R(A)$.

Function f

D2.8. A function is a relation such that no two of its elements have the same 1^{st} coordinate ($\forall a, b, c\,[(\langle a, b\rangle \in f \wedge \langle a, c\rangle \in f) \Rightarrow b = c]$).

- $f : A \to B$ if $\operatorname{dom}(f) = A$ and $\operatorname{ran}(f) \subseteq B$.

F2.9. If f is a function and $g \subseteq f$, then g is a function.
F2.11. If f is a function and A is any collection, then $f \restriction A$ is also a function.

- If $A \subseteq \operatorname{dom}(f)$, then $\operatorname{dom}(f \restriction A) = A$

D2.21. $X^Y = \{f : f \text{ is a function} \wedge f : Y \to X\}$.

Inverse of Function f^{-1}

D2.14. If f is a function and B is a collection, $f^{-1}(B) = \operatorname{Im}_{f^{-1}}(B) = \{a : \exists b \in B\,[\langle a, b\rangle \in f]\}$.
C2.17. Let f be any function and A and B be any collections of sets. Then:

- $f^{-1}(\bigcup A) = \bigcup\{I : \exists a \in A\,[I = f^{-1}(a)$.
- $f^{-1}(\bigcap A) = \bigcap\{I : \exists a \in A\,[I = f^{-1}(a)$.
- $f^{-1}(B \backslash A) = f^{-1}(B) \backslash f^{-1}(A)$.

Composite Function $g \circ f$

D2.18. f composed with g, $g \circ f = \{x : \exists a\, \exists b\, \exists c\,[(\langle a, b\rangle \in f) \wedge (\langle b, c\rangle \in g) \wedge (x = \langle a, c\rangle)]\}$.
L2.19. Let f, g, h be functions, then:

- $g \circ f$ is a function.
- If $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$.
- (*Associativity*) $h \circ (g \circ f) = (h \circ g) \circ f$.

Injection, Surjection and Bijection

D2.20. Let $f : A \to B$ be a function, then:

- (*1-1*) $\forall a, a' \in A\,[f(a) = f(a') \Rightarrow a = a']$.
- (*onto*) $\operatorname{ran}(f) = B$.
- (*bijective*) *1-1* and *onto*.

L2.22. If $f : A \to B$ is *1-1* and *onto* B, then f^{-1} is *1-1* and *onto* A.

Directed Collection

D2.39. A collection G is called directed if

$$\forall a, b \in G\, \exists c \in G\,[a \subseteq c \wedge b \subseteq c]$$

L2.40. Let G be a directed collection of functions, then $f = \bigcup G$ is a function. Moreover, $\operatorname{dom}(f) = \bigcup\{\operatorname{dom}(\sigma) : \sigma \in G\}$ and $\operatorname{ran}(f) = \bigcup\{\operatorname{ran}(\sigma) : \sigma \in G\}$.

Cartesian Product $\prod$

Conv. A function f such that $\forall f \in I = \operatorname{dom}(f)\,[f(i) = A_i]$ is equivalent as a sequence $F = \langle A_i : i \in I\rangle$.
D2.36. $\prod F = \{\text{func } f : \operatorname{dom}(f) = I \wedge \forall i \in I\,[f(i) \in A_i]\}$.
T2.46. (*General Distributive Laws*) Let I be a set and $\langle J_i : i \in I\rangle$ be a sequence of sets. Suppose that $I \neq \emptyset$ and $\forall i \in I\,[J_i \neq \emptyset]$. For each $i \in I$, let $\langle A_{i,j} : j \in J_i\rangle$ be a sequence of sets. Then:

$$\bigcup_{i\in I} \bigcap_{j\in J_i} A_{i,j} = \bigcap\{\bigcup_{i\in I} A_{i,f(i)} : f \in \prod_{i\in I} J_i\}$$
$$\bigcap_{i\in I} \bigcup_{j\in J_i} A_{i,j} = \bigcup\{\bigcap_{i\in I} A_{i,f(i)} : f \in \prod_{i\in I} J_i\}$$
$$\prod_{i\in I}(\bigcup_{j\in J_i} A_{i,j}) = \bigcup\{\prod_{i\in I} A_{i,f(i)} : f \in \prod_{i\in I} J_i\}$$
$$\prod_{i\in I}(\bigcap_{j\in J_i} A_{i,j}) = \bigcap\{\prod_{i\in I} A_{i,f(i)} : f \in \prod_{i\in I} J_i\}$$

T2.47. Fix $n \geq 1$. Let X be a set and let $A_1, A_2, \ldots, A_n$ be subsets of X. Then there are at most 2^{2^n} different sets that can be formed from $A_1, A_2, \ldots, A_n$ using the operations $X \backslash \cdot$, $\cup$ and $\cap$ (number of regions in a Venn diagram).

Russell's Paradox

T3.1. (*Russell*) $R = \{x : x \text{ is a set } \wedge x \notin x\}$ is not a set.
T3.3. $V = \{x : x \text{ is a set}\}$ is not a set.
E3.4. If A and B are sets, then $A \times B$ is also a set.

E3.5. If A and B are sets, then $\mathrm{dom}(A), \mathrm{ran}(A), \bigcap A, A^B$ are sets.
E3.6. I is a set and $\langle A_i : i \in I\rangle$ is a sequence. Then $\prod_{i\in I} A_i$ is a set.
E3.7. R and A are sets. If R is a relation, then $\mathrm{Im}_R(A)$ is a set.
E3.8. The class $\mathbf{U} = \{x : \exists a \exists b\, [x = \langle a, b\rangle]\}$ is a set.
E3.9. If f is a function and $\mathrm{dom}(f)$ is a set, then f is a set.
E3.x. $\mathbb{U} = \{A : A \text{ is a set and } \mathbb{N} \approx A\}$ is not a set. Suppose $\mathbb{U}$ is a set. Fix any $x \in \mathbb{V}$. Then x is a set, so $A_x = \{x\} \times \mathbb{N}$ is a set. $\mathbb{N} \approx A_x$ since $\exists f(n) = \langle x, n\rangle$ that is bijective. For any $x \in \mathbf{V}$, we have $x \in \{x\} \in \langle x, 0\rangle \in A_x \in \mathbb{U}$. Hence $\mathbf{V} \subseteq \bigcup\bigcup\bigcup \mathbf{U}$, contradiction.

The Natural Numbers

F4.1. (*Peano Axioms*) **L4.6** + **L4.7** + **L4.14** + **E4.15**(6)

Natural Number Set $\mathbb{N}$

D4.3. 0 is the empty set $\emptyset$.
D4.2. $S(x) = x \cup \{x\}$. $1 = S(0) = \{0\}$.
D4.4. A class A is called inductive if $0 \in A$ and $\forall x \in A\, [S(x) \in A]$. A set n is called a natural number if it belongs to every inductive class.

L4.6. $\begin{cases} 0 \in \mathbb{N} \\ n \in \mathbb{N} \Rightarrow S(n) \in \mathbb{N} \end{cases}$

L4.7. If X is any set of natural numbers such that $0 \in X$ and $\forall x \in X\, [S(x) \in X]$, then X is the set of all natural numbers.
F4.8. (*Principle of Mathematical Induction*) P is some property. Suppose that 0 has property P and $\forall n \in \mathbb{N}\, [n \text{ has property } P \Rightarrow S(n) \text{ has property } P]$. Then all natural numbers have property P.

L4.9. $\begin{cases} \forall x \in n\, [x \subseteq n] \\ n \subseteq \mathbb{N} \\ \forall x\, [(x \subseteq n \wedge x \neq \emptyset) \Rightarrow \exists m \in x\, [x \cap m = \emptyset]] \end{cases}$

L4.10. $\begin{cases} n \notin n \\ m \subseteq n \Rightarrow (m \in n \vee m = n) \\ (m \subseteq n \wedge n \in k) \Rightarrow m \in k \\ \text{Either } m = n \text{ or } m \in n \text{ or } n \in m. \end{cases}$

L4.11. Let $X \subseteq \mathbb{N}$. If $X \neq \emptyset$, then $\exists n \in X\, [X \cap n = \emptyset]$.
L4.14. $\forall n, m \in \mathbb{N}\, [n \neq m \Rightarrow S(n) \neq S(m)]$.

Less Than Relation $<$

D4.12. $\forall n, m \in \mathbb{N}\, [m < n \Leftrightarrow m \subset n]$.
F4.13. (*Principle of Strong Induction*) P is some property. Suppose that $\forall n \in \mathbb{N}$ [if P holds for all $m \in \mathbb{N}$ less than n, then P holds for n]. Then P holds for all $n \in \mathbb{N}$.

E4.15. $\begin{cases} m \in n \in k \Rightarrow m \in k \\ m \in n \in S(m) \text{ is impossible.} \\ n \neq 0 \Rightarrow n = S(\bigcup n) \\ n \leq m \Leftrightarrow n \subseteq m \\ \max\{n, m\} = n \cup m \\ \text{Either } n = 0 \text{ or } \exists k \in n\, [S(k) = n]. \end{cases}$

E4.16. $X \subseteq \mathbb{N}$. Suppose X has the property that $\forall n \in X\, [n \subseteq X]$, then either $X = \mathbb{N}$ or $\exists n \in \mathbb{N}\, [X = n]$.

Extender **E***, Addition + and Multiplication* $\cdot$

D4.17. Let **FN** denote the class of all functions whose domain is some natural number (**FN** is a proper class):

$$\mathbf{FN} = \{\sigma : \sigma \text{ is a function} \wedge \exists n \in \mathbb{N}\, [\mathrm{dom}(\sigma) = n]\}$$

An extender is a function $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$.
T4.19. Suppose $\mathbf{E} : \mathbf{FN} \to \mathbf{V}$ is any extender. Then $\exists! f : \mathbb{N} \to \mathbf{V}\, [\forall n \in \mathbb{N}\, [f(n) = \mathbf{E}(f \restriction n)]]$.
D4.25. Define $\mathbf{E}(\sigma) = \begin{cases} m & \mathrm{dom}(\sigma) = 0 \\ S(\sigma(\bigcup \mathrm{dom}(\sigma))) & \mathrm{dom}(\sigma) \neq 0 \end{cases}$.
$\exists! f_m$ corresponds to $\mathbb{E}$. Define $m + n = f_m(n)$.
Define $\mathbf{E}(\sigma) = \begin{cases} 0 & \mathrm{dom}(\sigma) = 0 \vee \sigma(\bigcup \mathrm{dom}(\sigma)) \notin \mathbb{N} \\ f_{\sigma(\bigcup \mathrm{dom}(\sigma))}(m) & \mathrm{dom}(\sigma) \neq 0 \wedge \sigma(\bigcup \mathrm{dom}(\sigma)) \in \mathbb{N} \end{cases}$.
$\exists! g_m$ corresponds to $\mathbb{E}$. Define $m \cdot n = g_m(n)$.
More generally, suppose we have a function $f : \mathbb{N} \to B$ that is defined recursively as $f(0) = b_0$ and $f(n+1) = h(f(n))$, then the extender corresponding to f should be defined as

$$\mathbb{E}(\sigma) = \begin{cases} b_0 & \mathrm{dom}(\sigma) = 0 \\ h(\sigma(\bigcup \mathrm{dom}(\sigma))) & \mathrm{dom}(\sigma) \neq 0 \wedge \sigma(\bigcup \mathrm{dom}(\sigma)) \in B \\ \emptyset & \mathrm{dom}(\sigma) \neq 0 \wedge \sigma(\bigcup \mathrm{dom}(\sigma)) \notin B \end{cases}$$

E4.26. $\begin{cases} n + 1 = S(n) \\ n + (m + k) = (n + m) + k \\ n + m = m + n \\ n + n = 2 \cdot n \\ 2 \cdot n = 2 \cdot m \Rightarrow n = m \\ n \cdot (m + k) = n \cdot m + n \cdot k \\ n \cdot (m \cdot k) = (n \cdot m) \cdot k \\ n \cdot m = m \cdot n \end{cases}$

E4.27. $\begin{cases} n < k \Rightarrow m + n < m + k \\ m \neq 0 \wedge n < k \Rightarrow m \cdot n < m \cdot k \end{cases}$

Set Sizes

D5.1. $A \approx B \Leftrightarrow \exists f : A \to B$ which is both *1-1* and *onto*.
F5.2. For any set A, $\mathcal{P}(A) \approx \{0, 1\}^A$.
D5.4. $A \precsim B$ if there exists $f : A \to B$ which is *1-1*.
L5.5. If f and g are both *1-1*, then $g \circ f$ is also *1-1*.

L5.6. $\begin{cases} A \precsim A \\ (A \precsim B \wedge B \precsim C) \Rightarrow (A \precsim C) \\ (A \approx B \wedge B \approx C) \Rightarrow (A \approx C) \end{cases}$

T5.7. (*Cantor*) For any set X, $X \precnapprox \mathcal{P}(X)$.
D5.12. (*Schröder-Bernstein*) $A \precsim B \wedge B \precsim A \Rightarrow A \approx B$.

E5.13. $f : X \to Y$ is a *1-1* function. Then $\forall Z \subseteq X\, [Z \approx \mathrm{Im}_f(Z)]$.
E5.14. $I \subseteq A$ and $J \subseteq B$. If $I \approx J$ and $(A \backslash I) \approx (B \backslash J)$, then $A \approx B$.

E5.15. $m, n \in \mathbb{N}$. $\begin{cases} f \text{ is } \textit{1-1} \Rightarrow f \text{ is } \textit{onto}. \\ m \in n \Rightarrow m \precnapprox n \\ x \subsetneq n \Rightarrow x \precnapprox n \\ n \precnapprox \mathbb{N} \\ (A \approx n \wedge B \approx m \wedge A \cap B = \emptyset) \Rightarrow (A \cup B \approx n + m) \end{cases}$

E5.16. If $n \in \mathbb{N}$ and $A \approx S(n)$, then $\forall a \in A\, [A \backslash \{a\} \approx n]$.
L5.20. Suppose A and B are sets and $f : A \to B$ is a *1-1* function. Then $\forall X, Y \subseteq A\, [\mathrm{Im}_f(X) = \mathrm{Im}_f(Y) \Rightarrow X = Y]$.

L5.21. $\begin{cases} A \precsim B \Rightarrow \mathcal{P}(A) \precsim \mathcal{P}(B) \\ A \precsim B \Rightarrow A^C \precsim B^C \\ (A \precsim B \wedge C \precsim D \wedge B \cap D = \emptyset) \Rightarrow A \cup C \precsim B \cup D \end{cases}$

L5.23. If $n \in \mathbb{N}$ and $\exists$ *onto* function $\sigma : n \to A$, then $A \precsim n$.

Finite Set

D5.19. A is finite if $\exists n \in \mathbb{N}\, [n \approx A]$, otherwise it is infinite. A is countable if $A \precsim \mathbb{N}$, otherwise it is uncountable.
L5.22. If $n \in \mathbb{N}$ and $A \precsim n$, then A is finite.
L5.24. If A and B are finite, then so is $A \cup B$.
T5.25. Let A be a finite set and f is a function with $\mathrm{dom}(f) = A$, then:

- $X \subsetneq A \Rightarrow X \precnapprox A$.
- $\mathrm{ran}(f)$ is finite and $\mathrm{ran}(f) \precsim A$.
- If $\forall a \in A\, [a \text{ is finite}]$, then $\bigcup A$ is finite.
- $\mathcal{P}(A)$ is finite.

E5.26. If A is a finite non-empty subset of $\mathbb{N}$, then $\max(A) = \bigcup A$.
E5.27. $(A \precsim C \wedge B \precsim D) \Rightarrow (A \times B \precsim C \times D)$.

- If A and B are finite, then $A \times B$ is finite.
- If A and B are finite, then A^B is finite.

E5.28. If I is finite and $\forall i \in I\, [A_i \text{ is finite}]$, then $\prod_{i\in I} A_i$ is finite.
E5.30. Suppose f is any function, then $\mathrm{dom}(f) \approx f$.

Legends

C	Corollary
D	Definition
E	Exercise
F	Fact
L	Lemma
T	Theorem
Conv.	Convention