MA3205 Set Theory

AY2022/23 Semester 1 · Prepared by Tian Xiao

Sets and Operations

Axiom of Extensibility

For any sets A and B, $A = B$ if and only if $\forall x, (x \in A) \Leftrightarrow (x \in B)$.

Quantifiers

$\neg$ | $\wedge$ | $\vee$ | $\Rightarrow$ | $\Leftrightarrow$ | $\forall$ | $\exists$

Empty Set

$\emptyset$ denotes *the* empty set.

Set Operations

$\cup$ | $\cap$ | $\setminus$ | $\triangle$ | $\mathcal{P}$

$\triangle$ represents *symmetric difference* (i.e. $x \triangle y = (x \setminus y) \cup (y \setminus x)$).

$\mathcal{P}$ represents *power set* (i.e. $\mathcal{P}(x) = \{z : z \subseteq x\}$).

Union and Intersection of Collection of Sets

$$\bigcup A = \{x : \exists y \, [(y \in A) \wedge (x \in y)]\}$$

$$\bigcap A = \begin{cases} \emptyset & \text{if } A = \emptyset, \\ \{x : \forall y \, [y \in A \Rightarrow x \in y]\} & \text{otherwise.} \end{cases}$$

Pairing, Products and Relations

Ordered Pairs

$$\langle a, b \rangle = \{\{a\}, \{a, b\}\}$$

Lemma 2.2

$\langle x, y \rangle = \langle a, b \rangle$ if and only if $x = a$ and $y = b$.

Cartesian Product of Sets

$$A \times B = \{z : \exists a \in A, b \in B \, [z = \langle a, b \rangle]\}$$

Relation

A relation is a collection of ordered pairs.

- Domain: $dom(R) = \{a : \exists b\,[\langle a, b\rangle \in R]\}$
- Range: $ran(R) = \{b : \exists a\,[\langle a, b\rangle \in R]\}$
- Inverse: $R^{-1} = \{\langle b, a\rangle : \langle a, b\rangle \in R\}$
- $R_{less} = \{\langle r, s\rangle \in \mathbb{R} \times \mathbb{R} : r < s\}$
- Restriction: $R \upharpoonright A = R \cap (A \times ran(R))$
- Image: $Im_R(A) = \{b : \exists a \in A\,[\langle a, b\rangle \in R]\}$

Function

A function is a relation that no two of its elements have the same first coordinate. Formally, f is a function if and only if f is a relation and $\forall a, b, c\,[(\langle a, b\rangle \in f) \wedge (\langle a, c\rangle \in f) \Rightarrow (b = c)]$. $f : A \to B$ if $dom(f) = A$ and $ran(f) \subseteq B$.

- Note that $Im_f(A)$ is different from $f(A)$. Suppose A is a set, $Im_f(A)$ represents the set of second coordinates of all the elements in A, whereas $f(A)$ represents the second coordinate of A itself.
- Composite function: $f \circ g = \{\langle a, c\rangle : \exists b\,[(\langle a, b\rangle \in g) \wedge (\langle b, c\rangle \in f)]\}$
- Injection: f is injective (1-1) if $\forall a, a' \in dom(f)\,[(f(a) = f(a')) \Rightarrow (a = a')]$.
- Surjection: $f : A \to B$ is surjective (f is onto B) if $ran(f) = B$.
- Bijection: f is bijective if it is both injective and surjective.
- $X^Y = \{f : (f \text{ is a function}) \wedge (f : Y \to X)\}$

Lemma 2.15

Let R be a relation and A be a collection, then $Im_R(\bigcup A) = \bigcup\{Im_R(a) : a \in A\}$.

Lemma 2.16

Let R be a relation such that for any $x \neq y$, $Im_R(\{x\}) \cap Im_R(\{y\}) = \emptyset$, then:

1. $Im_R(\bigcap A) = \bigcap\{Im_R(a) : a \in A\}$;
2. $Im_R(A \backslash B) = Im_R(A) \backslash Im_R(B)$.

Lemma 2.19

Let f, g, h be functions, then:

1. $g \circ f$ is a function;
2. If $f : A \to B$ and $g : B \to C$, then $g \circ f : A \to C$;
3. $h \circ (g \circ f) = (h \circ g) \circ f$.

Lemma 2.22

If $f : A \to B$ is 1-1 and onto B, then $f^{-1} : B \to A$ is 1-1 and onto A.

Sequence

In set theory, functions are sequences. Suppose F is function with $dom(F) = I$, then the function can be written as a sequence $F = \langle A_i : i \in I\rangle$.

Cartesian Product of a Function

Suppose $F = \langle A_i : i \in I \rangle$, then $\prod F = \{f : (f \text{ is a function}) \wedge (dom(f) = I) \wedge (\forall i \in I\ [f(i) \in A_i])\}$
.

Axiom of Choice

If $\langle A_i : i \in I \rangle$ is any sequence of sets such that $\forall i \in I\ [A_i \neq \emptyset]$, then $\prod\limits_{i \in I} A_i \neq \emptyset$.

Directed Collections

A collection G is called *directed* if $\forall a, b \in G, \exists c \in G\ [a \subseteq c \wedge b \subseteq c]$.

Lemma 2.40

Let G be a directed collection of functions, then $f = \bigcup G$ is a function. Moreover, $dom(f) = \bigcup\{dom(\sigma) : \sigma \in G\}$ and $ran(f) = \bigcup\{ran(\sigma) : \sigma \in G\}$.

Theorem 2.46 (AC)

Let I be a set and $\langle J_i : i \in I \rangle$ be a sequence of sets. Suppose that $I \neq \emptyset$ and that $\forall i \in I\ [J_i \neq \emptyset]$. For each $i \in I$, let $\langle A_{i,j} : j \in J_i \rangle$ be a sequence of sets. Then:

1. $\bigcup\limits_{i \in I} \bigcap\limits_{j \in J_i} A_{i,j} = \bigcap \left\{ \bigcup\limits_{i \in I} A_{i,f(i)} : f \in \prod\limits_{i \in I} J_i \right\}$
2. $\bigcap\limits_{i \in I} \bigcup\limits_{j \in J_i} A_{i,j} = \bigcup \left\{ \bigcap\limits_{i \in I} A_{i,f(i)} : f \in \prod\limits_{i \in I} J_i \right\}$
3. $\prod\limits_{i \in I} \left(\bigcup\limits_{j \in J_i} A_{i,j} \right) = \bigcup \left\{ \prod\limits_{i \in I} A_{i,f(i)} : f \in \prod\limits_{i \in I} J_i \right\}$
4. $\prod\limits_{i \in I} \left(\bigcap\limits_{j \in J_i} A_{i,j} \right) = \bigcap \left\{ \prod\limits_{i \in I} A_{i,f(i)} : f \in \prod\limits_{i \in I} J_i \right\}$

Theorem 2.47

X is a set with $A_1, A_2, \ldots, A_n$ as subsets of X. There are at most 2^{2^n} sets which can be formed by repeating $\cap$, $\cup$ and $X \backslash$.

Russell's Paradox and Proper Classes

Russell's Argument

Let $R = \{x : x \text{ is a set and } x \notin x\}$, then R is not a set.

Rules about Sets and Classes

1. Everything is a class.
2. Every set is a class. Every class is a collection of sets. A class is a set if and only if it is a member of some class.
3. Every collection of sets is a class.
4. If A is a class and x is a set, then $A \cap x$ is a set.
5. The image of a set under a function is a set.

6. If A and B are sets, then so are A, B, $\bigcup A$, and $\mathcal{P}(A)$.
7. Axiom of Choice
8. Axiom of Infinity: The collection of natural numbers is a set.
9. Axiom of Extensibility

Theorem 3.3

$V = \{x : x \text{ is a set}\}$ is not a set.

Modelling and Properties of $\mathbb{N}$

$0 = \phi$; $S(x) = x \cup \{x\}$

Foundation: Inductive Class A $\begin{cases} 0 \in A & ① \\ \forall x \in A,\ S(x) \in A & ② \end{cases}$

Natural number n: Member of every inductive class

$\mathbb{N} = \{n : n \text{ is a natural member}\}$

"set"

Axiom of Infinity

Define $\begin{cases} m < n \Leftrightarrow m \in n. \\ m \le n \Leftrightarrow m \subseteq n \end{cases}$

$\begin{cases} 0 \in \mathbb{N} \\ n \in \mathbb{N} \Rightarrow S(n) \in \mathbb{N} \end{cases}$

Peano Axiom 1

Peano Axiom 2

$\mathbb{N}$ is the "smallest" inductive class

$\boxed{\text{To prove } \forall n \in \mathbb{N},\ P(n) \text{ is true: Let } X = \{n \in \mathbb{N} : P(n) \text{ is true}\} \text{ and show } X \text{ satisfies ① and ②}}$

Lemma 4.9

Let $n \in \mathbb{N}$. $\begin{cases} \forall x \in n,\ x \subseteq n \\ n \subseteq \mathbb{N} \\ \forall x \subseteq n \\ [x \neq \phi \Rightarrow \exists m \in x\ (x \cap m = \phi)] \end{cases}$

Use this to prove by contradiction that $x = \phi$.

Lemma 4.10

$n, m, k \in \mathbb{N}$ $\begin{cases} n \notin n \quad n \in n \Rightarrow x = \{n\} \subseteq n \Rightarrow \begin{cases} x \cap n = \phi \\ n \in x \wedge n \in n \Rightarrow n \in x \cap n \end{cases} \\ m \subseteq n \Rightarrow (m \in n \vee m = n) \\ (m \subseteq n \wedge n \in k) \Rightarrow m \in k \end{cases}$

Either $m = n$, $m \in n$ or $n \in m$.

Lemma 4.11

If $X \subseteq \mathbb{N}$, $X \neq \phi$, then $\exists n \in X\ [n \cap X = 0]$

Pick $m \in X$. $\begin{cases} m \cap X = 0 \\ x = m \cap X \neq 0 \Rightarrow x \subseteq m,\ x \neq \phi \end{cases}$

$\Downarrow$

$m \in \mathbb{N},\ m \subseteq X$

$\exists n \in x\ [x \cap n = \phi]$

$\Downarrow \qquad \Downarrow$

$n \in X \quad x \cap n = \phi$

Principle of Strong Induction

$P(n)$ is a property of n. Then

$$\forall n \left[\forall m < n \, [P(m)] \Rightarrow P(n) \right] \Rightarrow \forall n \, [P(n)]$$

Lemma 4.14

Peano Axiom 3

$n, m \in \mathbb{N}$. If $n \neq m$, then $S(n) \neq S(m)$.

$\Downarrow$

$n \in m$ or $m \in n$

$S(n) = n \cup \{n\}$ $\quad$ $S(m) = m \cup \{m\}$

If $S(n) = S(m)$, $m \in S(n) \Rightarrow \begin{cases} m = n & \text{contradiction} \\ m \in n \end{cases}$

Contradiction

Definition 4.25

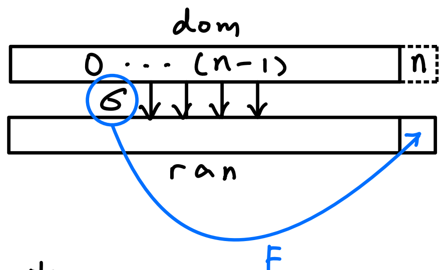
$$\text{"}+\text{"}: \begin{cases} m + 0 = m \\ m + S(n) = S(m+n) \end{cases}$$

Functions on $\mathbb{N}$

"proper class"

$$FN = \{ \sigma : \sigma \text{ is a function} \wedge \operatorname{dom}(\sigma) \in \mathbb{N} \}$$

Extender: $E: FN \to V$



Theorem 4.19

$E: FN \to V$ is any extender. Then there exists a unique function $f: \mathbb{N} \to V$ such that

$$\forall n \in \mathbb{N}, \; f(n) = E(f \restriction n)$$

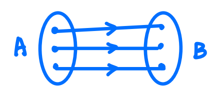
<u>Cardinality</u>

$$\{\overbrace{n,\ S(n),\ S(S(n)),\ S(S(S(n))),\ \cdots}^{\text{Cardinality}}\}$$

$\longrightarrow$ Ordinality

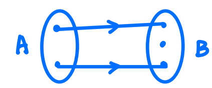
"the same size" $\boxed{A \approx B}$ "equinumerous"

$\exists f: A \to B$ s.t. f is 1-1 and onto.



$\downarrow$

"B is at least as big as A" $\boxed{A \lesssim B}$ $\exists f: A \to B$ s.t. f is 1-1.

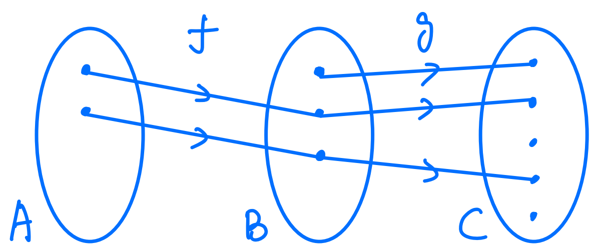


"B is strictly larger than A"

$\boxed{A \lnsim B}$ $\leftarrow$ $\wedge\ A \not\approx B$

$\exists f: A \to B$ s.t. f is 1-1 $\wedge$

$\neg \exists g: A \to B$ s.t. g is 1-1 and onto.



<u>Lemma 5.5</u>

$f: A \to B$ 1-1, $g: B \to C$ 1-1, then $g \circ f: A \to C$ is also 1-1.

<u>Lemma 5.6</u>

$$\begin{cases} A \lesssim A & f(a) = a \text{ is 1-1.} \\ (A \lesssim B \wedge B \lesssim C) \Rightarrow A \lesssim C \\ (A \approx B \wedge B \approx C) \Rightarrow A \approx C \end{cases}$$

<u>Theorem 5.7</u> Cantor

For any set X, $X \lnsim \mathcal{P}(X)$. $f(x) = \{x\} \in \mathcal{P}(X)$

$\rangle$ $\mathcal{P}(\mathbb{N})$ is uncountable.

Theorem 5.12 Schröder-Berstein

If $A \lesssim B$ and $B \lesssim A$, then $A \approx B$.

$\exists f: A \to B$ 1-1 $\exists g: B \to A$ 1-1

$\exists h: A \to B$ 1-1 and onto

> **Proof by Procrastination**
>
> Idea: Only make the decision you are forced to make.

Finite: $\exists n \in \mathbb{N},\ n \approx A$ $\longleftrightarrow$ Infinite

implies

Countable: $A \lesssim \mathbb{N}$ $\longleftrightarrow$ Uncountable

Lemma 5.20

counterpositive

Suppose $f: A \to B$ is 1-1. $\forall X, Y \in A$, if $\mathrm{Im}(X) = \mathrm{Im}(Y)$, then $X = Y$.

Lemma 5.21

① $A \lesssim B \Rightarrow \mathcal{P}(A) \lesssim \mathcal{P}(B)$

$\Downarrow$

$\exists f: A \to B$ 1-1   Define $G(X) = \mathrm{Im}_f(X)$ also 1-1

② $A \lesssim B \Rightarrow A^c \lesssim B^c$ Define $G(\sigma)(x) = f(\sigma(x))$

③ If $A \lesssim B$, $C \lesssim D$, $B \cap D = \phi$, then $A \cup C \lesssim B \cup D$.

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in C \setminus A \end{cases}$$