MA3205 Set Theory

AY2022/23 Semester 1 · *Prepared by Tian Xiao @snoidetx*

Set Sizes

D5.1. $A \approx B \Leftrightarrow \exists f : A \to B$ which is both *1-1* and *onto*.
F5.2. For any set A, $\mathcal{P}(A) \approx \{0,1\}^A$.
D5.4. $A \lessapprox B$ if there exists $f : A \to B$ which is *1-1*.
L5.5. If f and g are both *1-1*, then $g \circ f$ is also *1-1*.

L5.6. $\begin{cases} A \lessapprox A \\ (A \lessapprox B \wedge B \lessapprox C) \Rightarrow (A \lessapprox C) \\ (A \approx B \wedge B \approx C) \Rightarrow (A \approx C) \end{cases}$

T5.7. (*Cantor*) For any set X, $X \lnapprox \mathcal{P}(X)$.
D5.12. (*Schröder-Bernstein*) $A \lessapprox B \wedge B \lessapprox A \Rightarrow A \approx B$.
E5.13. $f : X \to Y$ is a *1-1* function. Then $\forall Z \subseteq X\, [Z \approx \mathrm{Im}_f(Z)]$.
E5.14. $I \subseteq A$ and $J \subseteq B$. If $I \approx J$ and $(A \backslash I) \approx (B \backslash J)$, then $A \approx B$.

E5.15. $m, n \in \mathbb{N}$. $\begin{cases} f \text{ is } \textit{1-1} \Rightarrow f \text{ is } \textit{onto}. \\ m \in n \Rightarrow m \lnapprox n \\ x \subsetneq n \Rightarrow x \lnapprox n \\ n \lnapprox \mathbb{N} \\ (A \approx n \wedge B \approx m \wedge A \cap B = \emptyset) \Rightarrow (A \cup B \approx n + m) \end{cases}$

E5.16. If $n \in \mathbb{N}$ and $A \approx S(n)$, then $\forall a \in A\, [A \backslash \{a\} \approx n]$.
L5.20. Suppose A and B are sets and $f : A \to B$ is a *1-1* function. Then $\forall X, Y \subseteq A\, [\mathrm{Im}_f(X) = \mathrm{Im}_f(Y) \Rightarrow X = Y]$.

L5.21. $\begin{cases} A \lessapprox B \Rightarrow \mathcal{P}(A) \lessapprox \mathcal{P}(B) \\ A \lessapprox B \Rightarrow A^C \lessapprox B^C \\ (A \lessapprox B \wedge C \lessapprox D \wedge B \cap D = \emptyset) \Rightarrow A \cup C \lessapprox B \cup D \end{cases}$

L5.23. If $n \in \mathbb{N}$ and $\exists$ *onto* function $\sigma : n \to A$, then $A \lessapprox n$.

Finite Set

D5.19. A is finite if $\exists n \in \mathbb{N}\, [n \approx A]$, otherwise it is infinite. A is countable if $A \lessapprox \mathbb{N}$, otherwise it is uncountable.
L5.22. If $n \in \mathbb{N}$ and $A \lessapprox n$, then A is finite.
L5.24. If A and B are finite, then so is $A \cup B$.
T5.25. Let A be a finite set and f is a function with $\mathrm{dom}(f) = A$, then:

- $X \subsetneq A \Rightarrow X \lnapprox A$.
- $\mathrm{ran}(f)$ is finite and $\mathrm{ran}(f) \lessapprox A$.
- If $\forall a \in A\, [a \text{ is finite}]$, then $\bigcup A$ is finite.
- $\mathcal{P}(A)$ is finite.

E5.26. If A is a finite non-empty subset of $\mathbb{N}$, then $\max(A) = \bigcup A$.
E5.27. $(A \lessapprox C \wedge B \lessapprox D) \Rightarrow (A \times B \lessapprox C \times D)$.

- If A and B are finite, then $A \times B$ is finite.
- If A and B are finite, then A^B is finite.

E5.28. If I is finite and $\forall i \in I\, [A_i \text{ is finite}]$, then $\prod_{i \in I} A_i$ is finite.
E5.30. Suppose f is any function, then $\mathrm{dom}(f) \approx f$.

Orders

Quasi Order $\leq$

D6.2. $\leq$ is a quasi order on X if
(1) $\forall x \in X\, [x \leq x]$;
(2) $\forall x, y, z \in X\, [(x \leq y \wedge y \leq z) \Rightarrow (x \leq z)]$.

Partial Order $<$

D6.4. $<$ is a partial order on X if
(1) $\forall x \in X\, [x \not< x]$;
(2) $\forall x, y, z \in X\, [(x < y \wedge y < z) \Rightarrow (x < z)]$.

Linear Order $\lhd$

D6.5. $\lhd$ is a linear order on X if
(1) $\forall x \in X\, [x \ntriangleleft x]$;
(2) $\forall x, y, z \in X\, [(x \lhd y \wedge y \lhd z) \Rightarrow (x \lhd z)]$;
(3) $\forall x, y \in X\, [(x = y) \vee (x \lhd y) \vee (y \lhd x)]$.

Maximal and Minimal

D6.9. If $\langle X, < \rangle$ is a partial order, then $x \in X$ is a maximal element of X if $\forall y \in X\, [x \not< y]$. $x \in X$ is a minimal element of X if $\forall y \in X\, [y \not< x]$.
L6.10. Suppose $\langle X, < \rangle$ is a finite non-empty partial order, then X has both a maximal and a minimal element.

Chain and Antichain

D6.11. Let $\langle X, < \rangle$ be a finite partial order. A set $C \subseteq X$ is called a chain if $\forall x, y \in C\, [x \text{ and } y \text{ are comparable}]$. A set $A \subseteq X$ is called a chain if $\forall x \neq y \in A\, [x \text{ and } y \text{ are incomparable}]$.

- An antichain $A \subseteq X$ is maximal if there is no antichain $A' \subseteq X$ such that $A \subset A'$. A chain $C \subseteq X$ is maximal if there is no chain $C' \subseteq X$ such that $C \subset C'$.
- $\emptyset$ and singletons are both a chain and an antichain.

L6.12. Let $\langle X, < \rangle$ be a finite partial order. Every antichain in X is contained in a maximal antichain. Every chain in X is contained in a maximal chain.

Well Order

D6.13. A linear order $\langle X, < \rangle$ is called a well order if every non-empty subset of X has a minimal element.
L6.15. A linear order $\langle X, < \rangle$ is a well order if and only if there is no function $f : \mathbb{N} \to X$ such that $\forall n \in \mathbb{N}\, [f(n) > f(n+1)]$.

Predecessor pred

D6.16. Let $\langle X, < \rangle$ be a linear order. For any $x \in X$ define $\mathrm{pred}_{\langle X, < \rangle}(x) = \{x' \in X : x' < x\}$.

- A subset $A \subseteq X$ is downwards closed if $\forall a \in A \forall x \in X\, [x < a \Rightarrow x \in A]$.

F6.17. Let $\langle X, < \rangle$ be a linear order. Suppose $A \subseteq X$ is downwards closed. Then $\forall a \in A\, [\mathrm{pred}_{\langle A, < \rangle}(a) = \mathrm{pred}_{\langle X, < \rangle}(a)]$.
F6.19. Let $\langle X, < \rangle$ be a well order and let $A \subseteq X$ be downwards closed. Then either $A = X$ or $\exists x \in X\, [A = \mathrm{pred}_{\langle X, < \rangle}(x)]$.
E6.20. Let $\langle X, < \rangle$ be a well order and $A \subseteq X$. Then $\langle A, < \rangle$ is a well order.
E6.21. Let $\langle X, < \rangle$ be a linear order. We say that $f : X \to X$ is expansive if $\forall x \in X\, [f(x) \geq x]$. We say that $f : X \to X$ is order-preserving if $\forall x, y \in X\, [x < y \Rightarrow f(x) < f(y)]$. Prove that if $\langle X, < \rangle$ is a well order, then every order-preserving $f : X \to X$ is expansive.

New Order from the Old

L6.23. Suppose that X is a set and that $\langle Y, \prec \rangle$ and $\langle Z, \lhd \rangle$ are partial orders. Suppose $f : X \to Y$ and $g : X \to Z$ are any functions. Define $<$ on X by stipulating that for $x, x' \in X$,

$$x < x' \Leftrightarrow (f(x) \prec f(x')) \vee [(f(x) = f(x')) \wedge (g(x) \lhd g(x'))].$$

Then the following hold:
(1) $<$ is a partial order on X;
(2) If $\langle Y, \prec \rangle$ and $\langle Z, \lhd \rangle$ are both linear orders and $\forall x, x' \in X\, [(f(x) = f(x') \wedge g(x) = g(x') \Rightarrow x = x']$, then $<$ is a linear order on X.
(3) If $\langle Y, \prec \rangle$ and $\langle Z, \lhd \rangle$ are both well orders and $\forall x, x' \in X\, [(f(x) = f(x') \wedge g(x) = g(x') \Rightarrow x = x']$, then $<$ is a well order on X.

C6.24. Let X be a set and $\langle Y, \prec \rangle$ be a partial order. Suppose $f : X \to Y$ is any function. Define $<^*$ on X by stipulating that for any $x, x' \in X$,

$$x <^* x' \Leftrightarrow f(x) \prec f(x')$$

Then $<^*$ is a partial order on X. Furthermore, if f is *1-1* and $\prec$ is a linear order on Y, then $<^*$ is a linear order on X. If f is *1-1* and $\prec$ is a well order on Y, then $<^*$ is a well order on X.
D6.26. Suppose $\langle I, < \rangle$ is any well order and X is any set. For $f, g \in X^I$, if $f \neq g$, define

$$\Delta(f, g) = \min(\langle \{i \in I : f(i) \neq g(i)\}, < \rangle).$$

L6.27. Suppose $\langle X, \lhd \rangle$ is a linear order and $\langle I, < \rangle$ is a well order. Define a relation $\prec$ on X^I by stipulating that for any $f, g \in X^I$,

$$f \prec g \Leftrightarrow f \neq g \wedge f(\Delta(f, g)) \lhd g(\Delta(f, g)).$$

Then $\prec$ is a linear order on X^I.
D6.28. For each $n \in \mathbb{N}$,

$$\begin{cases} [\mathbb{N}]^n = \{a \in \mathcal{P}(\mathbb{N}) : a \approx n\} \\ [\mathbb{N}]^{<\omega} = \{a \in \mathcal{P}(\mathbb{N}) : a \text{ is finite}\} = \bigcup_{n \in \mathbb{N}} [\mathbb{N}]^n \\ \mathbb{N}^n = \{\sigma : \sigma \text{ is a function} \wedge \mathrm{dom}(\sigma) = n \wedge \mathrm{ran}(\sigma) = \mathbb{N}\} \\ \mathbb{N}^{<\omega} = \{\sigma : \sigma \text{ is a function} \wedge \mathrm{dom}(\sigma) \in \mathbb{N} \wedge \mathrm{ran}(\sigma) = \mathbb{N}\} = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n \end{cases}$$

E6.31. $\mathbb{N}^{\mathbb{N}}$ is dense. $2^{\mathbb{N}}$ is not dense.

Embeddings and Isomorphisms

D6.33. If $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ are linear orders, then a function $f : X \to Y$ is an isomorphism between $\langle X, \lhd \rangle$ and $\langle Y, \prec \rangle$ if the following hold:

(1) f is *1-1* and *onto*;

(2) $\forall x, y \in X\,[x \lhd y \Leftrightarrow f(x) \prec f(y)]$.

L6.34. $\langle X, \lhd\rangle$ and $\langle Y, \prec\rangle$ are linear orders. Suppose $f : X \to Y$ is an *onto* function such that $\forall x, y \in X\,[x \lhd y \Rightarrow f(x) \prec f(y)]$. Then f is an isomorphism.

D6.35. $\langle X, <\rangle$ and $\langle Y, \prec\rangle$ are linear orders. A function $f : X \to Y$ is called an embedding if $\forall x, x' \in X\,[x < x' \leftrightarrow f(x) \prec f(x')]$ and f is *1-1*. If there exists such embedding, $\langle X, <\rangle \hookrightarrow \langle Y, \prec\rangle$.

F6.36. $\langle X, <\rangle$ and $\langle Y, \prec\rangle$ are linear orders. If $f : X \to Y$ is a function such that $\forall x, x' \in X\,[x < x' \Rightarrow f(x) \prec f(x')]$, then f is an embedding.

F6.37. $\langle X, <\rangle$ and $\langle Y, \prec\rangle$ are linear orders. Suppose A and B are downwards closed subsets of X and Y respectively. If $f : A \to B$ is an isomorphism from $\langle A, <\rangle$ to $\langle B, \prec\rangle$, then for any $a \in A$, $f \restriction \mathrm{pred}_{\langle X,<\rangle}(a)$ is an isomorphism from $\langle \mathrm{pred}_{\langle X,<\rangle}(a), <\rangle$ to $\langle \mathrm{pred}_{\langle Y,\prec\rangle}(f(a)), \prec\rangle$.

T6.38. Suppose $\langle X, \lhd\rangle$ is a finite linear order. Then there exists a unique $n \in \mathbb{N}$ such that $\langle X, \lhd\rangle$ is isomorphic to $\langle n, \in\rangle$. Moreover, this isomorphism is unique.

T6.39. Suppose $\langle X, \lhd\rangle$ is an infinite linear order such that for each $x \in X$, $\mathrm{pred}_{\langle X,\lhd\rangle}(x)$ is finite. Then $\langle X, \lhd\rangle$ is isomorphic to $\langle \mathbb{N}, \in\rangle$. Moreover, this isomorphism is unique.

D6.42. A linear order $\langle X, \lhd\rangle$ has *type omega* if X is infinite and for every $x \in X$, $\mathrm{pred}_{\langle X,\lhd\rangle}(x)$ is finite.

Countable and Uncountable Sets

C7.1. Suppose $X \subseteq \mathbb{N}$ is infinite. Then $\langle X, \in\rangle$ is isomorphic to $\langle \mathbb{N}, \in\rangle$.

C7.2. Suppose $X \subseteq \mathbb{N}$ is infinite and countable, then $X \approx \mathbb{N}$.

T7.3. There exists linear orders of *type omega* on the following objects: (1) $\mathbb{N} \times \mathbb{N}$, (2) $[\mathbb{N}]^{<\omega}$, (3) $\mathbb{N}^{<\omega}$.

C7.4. $\begin{cases} \mathbb{N} \times \mathbb{N} \approx \mathbb{N}; [\mathbb{N}]^{<\omega} \approx \mathbb{N}; \mathbb{N}^{<\omega} \approx \mathbb{N}. \\ \forall n \in \mathbb{N}\,[n \geq 1 \Rightarrow (\mathbb{N}^n \approx \mathbb{N} \wedge [\mathbb{N}]^n \approx \mathbb{N})]. \end{cases}$

L7.5. $\langle A_n : n \in \mathbb{N}\rangle$ and $\langle f_n : n \in \mathbb{N}\rangle$ are sequences such that for each $n \in \mathbb{N}$, $f_n : A_n \to \mathbb{N}$ is *1-1*. Then $\bigcup_{n\in\mathbb{N}} A_n$ is countable.

L7.6. A countable union of countable sets is countable.

L7.8. The set of rational numbers $\mathbb{Q}$ is countable.

F7.12. If $x, y \in \mathbb{R}$ and $x < y$, there is a $q \in \mathbb{Q}$ with $x < q < y$.

L7.13. $2^{\mathbb{N}} \lessapprox \mathbb{R} \lessapprox \mathcal{P}(\mathbb{Q})$.

T7.14. $2^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}} \approx \mathcal{P}(\mathbb{N}^{\mathbb{N}}) \approx \mathcal{P}(\mathbb{N}) \approx \mathcal{P}(\mathbb{Q}) \approx \mathbb{R}$.

D7.15. A set X is set to be have size continuum or size $\mathfrak{c}$ if $X \approx \mathcal{P}(\mathbb{N})$.

L7.16. Let $r, s \in \mathbb{R}$ with $r < s$. Then (r, s) has size $\mathfrak{c}$, where $(r, s) = \{x \in \mathbb{R} : r < x < s\}$.

E7.17. Let $l \subseteq \mathbb{R}^2$ be a line, then $l \approx \mathbb{R}$.

L7.21. If $A \lessapprox B$ and $A \neq \emptyset$, then there exists an *onto* function $g : B \to A$.

L7.22. Suppose A and B are sets and $f : B \to A$ is *onto*, then $A \lessapprox B$.

L7.23. Suppose A, B and C are sets and $f : C \to B$ is *onto*, then $A^B \lessapprox A^C$.

C7.24. If $B \approx C$, then $A^B \approx A^C$.

C7.25. If $A \lessapprox D$ and $B \lessapprox C$, and $B \neq \emptyset$, then $A^B \lessapprox D^C$.

L7.26. There exists a sequence $\langle A_n : n \in \mathbb{N}\rangle$ of pairwise disjoint infinite subsets of $\mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{N}$.

L7.27. Suppose A, B, C are sets with $B \cap C = \emptyset$, then $A^B \times A^C \approx A^{B\cup C}$.

C7.28. $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}$. Hence $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$.

C7.29. $\mathbb{R}^2$ has size $\mathfrak{c}$.

L7.31. Let A, B, C be sets. $A^{B\times C} \approx (A^B)^C$.

C7.32. $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \approx \mathbb{N}^{\mathbb{N}}$.

C7.33. $\mathbb{R}^{\mathbb{N}}$ has size $\mathfrak{c}$.

D7.34. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous if for each $x \in \mathbb{R}$ and each $\epsilon > 0$, there exists $\delta > 0$ such that $\mathrm{Im}_f((x - \delta, x + \delta)) \subseteq (f(x) - \epsilon, f(x) + \epsilon)$. A set $U \subseteq \mathbb{R}$ is called an open interval if there exist $r, s \in \mathbb{R}$ such that $U = (r, s) = \{x \in \mathbb{R} : r < x < s\}$. $U \subseteq \mathbb{R}$ is called open if it is the union of a collection of open intervals.

L7.35. There are only $\mathfrak{c}$ many continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

L7.36. There are only $\mathfrak{c}$ many open subsets of $\mathbb{R}$.

E7.37. $\mathbb{R}^{\mathbb{R}} \approx 2^{\mathbb{R}}$.

E7.38. $\mathbb{Z}(X) \setminus \{0\} \approx \mathbb{N}$. There are countably many algebraic real numbers.

More about Partial and Linear Orders

T8.1. Suppose $\langle X, <\rangle$ is a finite partial order. Let $k(X) = \max\{m \in \mathbb{N} : \exists A \subseteq X\,[A \text{ is an antichain in } X \wedge A \approx m]\}$. Then X is a union of $k(X)$ disjoint chains.

D8.4. Suppose $\langle X, <\rangle$ is a finite partial order and $A \subseteq X$.

$$\begin{cases} \text{Upper bound } x\text{: } \forall a \in A\,[a \leq x]. \\ \text{Lower bound } x\text{: } \forall a \in A\,[x \leq a]. \\ \text{Supremum } u\text{: } \forall x \in \{\text{upper bounds}\}\,[u \leq x]. \\ \text{Infimum } u\text{: } \forall x \in \{\text{lower bounds}\}\,[x \leq u]. \end{cases}$$

D8.10. Let $\langle X, <\rangle$ be a linear order. A pair $\langle A, B\rangle$ is called a cut of $\langle X, <\rangle$ if the following hold:

(1) A is downwards closed.

(2) B is upwards closed.

(3) A and B partition X.

F8.11. Let $\langle X, <\rangle$ be a linear order and $Y \subseteq X$. If $z \in X \setminus Y$ and if $A = \{a \in Y : a < z\}$ and $B = \{b \in Y : z < b\}$, then $\langle A, B\rangle$ is a cut of $\langle Y, <\rangle$.

D8.13. A linear order $\langle X, <\rangle$ is called dense if $\forall x, y \in X\,\exists z \in X\,[x < y \Rightarrow x < z < y]$.

D8.14. A linear order $\langle X, <\rangle$ is without endpoints or has no endpoints if $\langle X, <\rangle$ has neither a maximal element or a minimal element.

T8.15. Suppose $\langle X, <\rangle$ is a non-empty dense linear order without endpoints. Let $\langle Y, \prec\rangle$ be any countable linear order. Then $\langle Y, \prec\rangle \hookrightarrow \langle X, <\rangle$.

T8.16. Let $\langle X, <\rangle$ and $\langle Y, \prec\rangle$ be any non-empty countable dense linear orders without endpoints. Then $\langle X, <\rangle$ and $\langle Y, \prec\rangle$ are isomorphic.

E8.19. The two countable linear orders $\langle (0, 1), <\rangle$ and $\langle [0, 1], <\rangle$ embed into each other but are not isomorphic.

E8.20. The two countable linear orders $\langle \mathbb{N}, \in\rangle$ and $\langle \mathbb{N}, \ni\rangle$ do not embed to each other.

Well Ordered Sets

F9.1. If $\langle X, <\rangle$ is a linear order of *type omega*, then $\langle X, <\rangle$ is a well order.

L9.2. Suppose $\langle X, <\rangle$ is a well order. Suppose A and B are downwards closed subsets of X. If $\langle A, <\rangle$ is isomorphic to $\langle B, <\rangle$, then $A = B$.

C9.3. Suppose $\langle X, <\rangle$ is a well order. Suppose $x < x' \in X$. Then $\langle \mathrm{pred}_{\langle X,<\rangle}(x'), <\rangle$ is not isomorphic to $\langle \mathrm{pred}_{\langle X,<\rangle}(x), <\rangle$.

C9.4. Suppose $\langle X, <\rangle$ is a well order. Then for any $x \in X$, $\langle \mathrm{pred}_{\langle X,<\rangle}(x), <\rangle$ is not isomorphic to $\langle X, <\rangle$.

L9.5. If $\langle X, <\rangle$ and $\langle Y, \prec\rangle$ are isomorphic well orders, then the isomorphism between them is unique.

T9.6. If $\langle X, <\rangle$ and $\langle Y, \prec\rangle$ are isomorphic well orders, then exactly one of the 3 followings hold:

(1) $\langle X, <\rangle$ is isomorphic to $\langle Y, \prec\rangle$;

(2) $\exists x \in X\,[\langle \mathrm{pred}_{\langle X,<\rangle}(x), <\rangle$ is isomorphic to $\langle Y, \prec\rangle]$;

(3) $\exists y \in Y\,[\langle \mathrm{pred}_{\langle Y,\prec\rangle}(y), \prec\rangle$ is isomorphic to $\langle X, <\rangle]$.

Legends

C	Corollary
D	Definition
E	Exercise
F	Fact
L	Lemma
T	Theorem
Conv.	Convention