MA3220 Ordinary Differential Equations

AY2022/23 Semester 1 · *Prepared by Tian Xiao @snoidetx*

1 First Order Equation

Separable Equation

Solve the separable ODE $y'(x) = P(x)Q(y)$.
The solution is $\int \frac{1}{Q(y)} dy = \int P(x)dx + C$.

Linear Equation (Existence & Uniqueness Theorem)

An ODE is *linear* if it is in the form $a_n(x)y^{(n)} + \cdots + a_1 y = P(x)$. It is *homogeneous* if $P(x) = 0$.

- p and g are cont. on $I = (\alpha, \beta)$ containing x_0. $\forall y_0 \in \mathbb{R}$, $\exists$ unique solution to $y' + p(x)y = g(x)$ for each x in I, with IC $y(x_0) = y_0$.

Solve the 1st order linear ODE $y' + P(x)y = Q(x)$.
Let $\mu(x) = e^{\int P(x)dx}$. The solution is $y = \frac{\int \mu(x)Q(x)dx}{\mu(x)y} + C$.

Nonlinear Equation (Existence & Uniqueness Theorem)

f and $\frac{\partial f}{\partial y}$ are both cont. in some rectangle $R = (\alpha, \beta) \times (\gamma, \delta)$ containing (x_0, y_0). In some interval $x_0 - h < x < x_0 + h$ contained in $\alpha < x < \beta$, $\exists$ unique solution to the IVP $y' = f(x, y)$ with IC $y(x_0) - y_0$.

Exact Equation

An ODE $M(x,y) + N(x,y)y' = 0$ is *exact* if $\exists \psi(x,y)\ [\psi_x = M \wedge \psi_y = n]$.

- If an ODE is exact, $M_y = N_x$.
- If M, N, M_y, N_x are cont. in a simply connected region $D \subset \mathbb{R}^2$, then the equation is exact if and only if $M_y = N_x$.

Solve the 1st order exact ODE $M(x,y) + N(x,y)y' = 0$.

$$\psi(x,y) = \int M(x,y)dx + g(y).$$

Find $g(y)$ by $\psi_y = N(x,y)$. The solution is $\psi(x,y) = C$.

Equilibrium Solution

$\leftarrow \cdot \rightarrow$	$\rightarrow \cdot \leftarrow$	$\leftarrow \cdot \leftarrow \quad \rightarrow \cdot \rightarrow$
unstable	asympt. stable	semi-stable

Euler's Method

$y_i = y_{i-1} + y'_{i-1} \times h$.

2 Second Order Equation

p, q, g are cont. on an open interval $I : \alpha < t < \beta$ containing t_0, $\exists$ unique solution to the IVP $y'' + p(t)y' + q(t)y = g(t)$ for each t in I, with IC $y(t_0) = y_0$ and $y'(t_0) = y'_0$. (Existence & Uniqueness Theorem)

Linear Homogeneous Equation

Superposition Principle y_1, y_2 solutions $\Rightarrow c_1 y_1 + c_2 y_2$ also solution.

Wronskian: $W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1 y'_2 - y_2 y'_1$.

- y_1, y_2 form a fundamental set if $\exists t_0 \in I\ [W[y_1, y_2](t_0) \neq 0]$.

Abel's Theorem Let y_1, y_2 be two solutions of $y'' + p(t)y' + q(t) = 0$ with p, q cont. in I. Then $W[y_1, y_2](t) = ce^{-\int p(t)dt}$. As a result, the Wronskian is either always 0 or never 0.

Solve the 2nd order ODE $ay'' + by + c = 0$.
Characteristic equation: $ar^2 + br + c = 0$.
Case I Distinct real roots r_1, r_2: $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
Case II Repeated real root r: $y = (c_1 + c_2 t)e^{rt}$.
Case III Complex roots $\lambda \pm \mu i$: $y = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t))$.

Given one solution $y_1(t)$ of the ODE $y''(t) + p(t)y'(t) + q(t)y(t) = 0$, find another solution $y_2(t)$.
Method I Abel's Theorem: Solve y_2 from $y_1 y'_2 - y_2 y'_1 = e^{-\int p(t)dt}$.
Method II Reduction of order. Let $y_2(t) = v(t)y_1(t)$ and solve $y_1(t)v''(t) + (2y'_1(t) + p(t)y_1(t))v'(t) = 0$. Solve by letting $u(t) = v'(t)$.

Linear Nonhomogeneous Equation

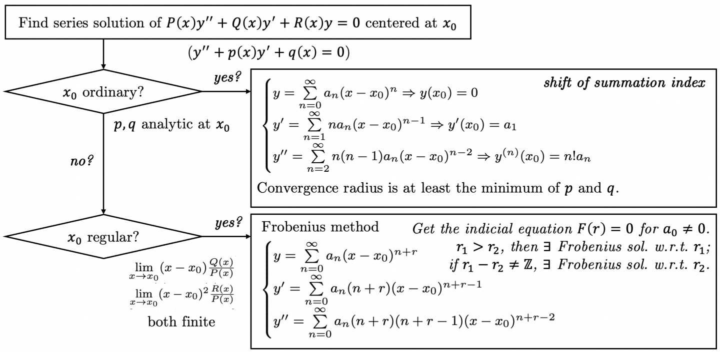
Find the particular solution of the ODE $ay'' + by' + cy = g(t)$.
Make the right guess: degree-n polynomial $\rightarrow$ degree-n polynomial; $Ce^{kt} \rightarrow Ae^{kt}$; $C\sin(kt)$ or $C\cos kt \rightarrow A\sin(kt) + B\cos(kt)$; sum/product of terms $\Rightarrow$ sum/product of their respective guesses. We handle exceptions by multiplying t to our guess when our guess solves the corresponding homogeneous equation (sum $\Rightarrow$ do for each term; product $\Rightarrow$ multiply the whole).

Find the general solution of the ODE $y'' + p(t)y' + q(t)y = g(t)$.
Variation of parameters: $\begin{cases} u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}\, dt + c_1 \\ u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}\, dt + c_2 \end{cases}$.
The general solution is $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$.

Power Series Solution

Ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x - x_0| < 1 \Rightarrow$ converge; $> 1 \Rightarrow$ diverge.
Convergence radius: $|x - x_0| > \rho \Rightarrow$ diverge; $< \rho \Rightarrow$ converge.

- If f polynomial, convergence radius of power series of $\frac{1}{f(x)}$ centered at x_0 = distance between x_0 and the nearest complex roots of $f(x)$.
- f, g analytic at t_0, radius $= \rho \Rightarrow fg, f + g$ analytic at t_0, radius $= \rho$.



Solve the Euler equation $(x - x_0)^2 y'' + \alpha(x - x_0)y' + \beta y = 0$.
Indicial equation: $F(r) = r^2 + (\alpha - 1)r + \beta = 0$.
Case I Distinct real roots r_1, r_2: $y = c_1|x - x_0|^{r_1} + c_2|x - x_0|^{r_2}$.
Case II Repeated real root r: $y = c_1|x - x_0|^r + c_2|x - x_0|^r \ln|x - x_0|$.
Case III Complex roots $\lambda \pm \mu i$: $y = |x - x_0|^{\lambda}(c_1 \cos(\mu \ln|x - x_0|) + c_2 \sin(\mu \ln|x - x_0|))$.

3 System of Equations

A general 1st order system is in the form $\begin{cases} x'_1(t) = F_1(t, x_1, \cdots, x_n) \\ \cdots \\ x'_n(t) = F_n(t, x_1, \cdots, x_n) \end{cases}$.

- Autonomous: Every F_i only depends on $x_1, \cdots, x_n$ and not t.
- Linear: Every F_i is linear: $F_i = p_{i1}(t)x_1 + \cdots + p_{in}(t)x_n + g_i(t)$.
- Homogeneous: The system is linear and every $g_i(t) \equiv 0$.
- Matrix form of linear system: $\mathbf{x}'(t) = \mathbf{P}(t)x + \mathbf{g}(t)$.

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \vdots & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

- All components of $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are cont. in some open interval I, $t_0 \in I$, then $\exists$ unique solution $\mathbf{x}(t)$ to the IVP $\mathbf{x}'(t) = \mathbf{P}(t)x + \mathbf{g}(t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ for all $t \in I$. (Existence & Uniqueness Theorem)

Linear Homogeneous System of Equations

Superposition Principle Consider the system $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}$, if $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$ are both solutions, then their linear combination $C_1\mathbf{x_1}(t) + C_2\mathbf{x_2}(t)$ is also a solution for any constant C_1, C_2.
General solution: If every solution can be written in some linear combination of $\mathbf{x_1}(t)$ and $\mathbf{x_2}(t)$, then they form a *fundamental set of solutions*. $C_1\mathbf{x_1}(t) + C_2\mathbf{x_2}(t)$ is the *general solution*.
Wronskian: $W[\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}](t) = \det\Big[\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)\Big]$.

- $\mathbf{P}(t)$ is cont. in (α, β). If $\exists t_0 \in (\alpha, \beta)\ \Big[W[\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}](t_0) \neq 0\Big]$, then $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ form a fundamental set of solutions for $t \in (\alpha, \beta)$.

Abel's Theorem In (α, β), W is either identically zero or never zero.

Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$. $\mathbf{A}$ has distinct eigenvalues.
If $\mathbf{A}$ has n distinct eigenvalues $r_1, \cdots, r_n$ and corresponding linearly independent eigenvectors $\mathbf{v}^{(1)}, \cdots, \mathbf{v}^{(n)}$, then the general solution is:

$$\mathbf{x}(t) = C_1\mathbf{v}^{(1)}e^{r_1 t} + \cdots + C_n\mathbf{v}^{(n)}e^{r_n t}.$$

If $\mathbf{A}$ has a complex eigenvalue $r = \lambda + \mu i$ and corresponding eigenvector $\mathbf{a} + \mathbf{b}i$, then the fundamental set of solutions is:

$$\mathbf{x}^{(1)}(t) = e^{\lambda t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t));$$
$$\mathbf{x}^{(2)}(t) = e^{\lambda t}(\mathbf{b}\cos(\mu t) + \mathbf{a}\sin(\mu t)).$$

Fundamental matrix: For $\mathbf{x}'(t) = \mathbf{P}(t)x$, suppose $\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)$ form a fundamental set of solutions. Then the $n \times n$ matrix $\mathbf{\Psi}(t) = [\mathbf{x}^{(1)}(t), \cdots, \mathbf{x}^{(n)}(t)]$ is called a *fundamental matrix* for the system.

- $\mathbf{\Psi}'(t) = \mathbf{P}(t)\mathbf{\Psi}(t)$.
- General solution: $\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{c}$.
- Solution to IVP with $\mathbf{x}(t_0) = \mathbf{x}_0$: $\mathbf{x}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1}\mathbf{x}_0 = \mathbf{\Phi}(t)\mathbf{x}_0$.
- The matrix $\mathbf{\Phi}(t) = \mathbf{\Psi}(t)\mathbf{\Psi}(t_0)^{-1}$ is also a fundamental matrix and it satisfies $\mathbf{\Phi}(t_0) = \mathbf{I}$.
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Matrix exponential: $e^{\mathbf{B}} = \mathbf{I} + \mathbf{B} + \frac{\mathbf{B}^2}{2!} + \cdots + \frac{\mathbf{B}^n}{n!} + \cdots$.

- $e^{\mathbf{A}\mathbf{t}} = \mathbf{I} + \mathbf{A}\mathbf{t} + \frac{\mathbf{A}^2 t^2}{2!} + \cdots + \frac{\mathbf{A}^n t^n}{n!} + \cdots$.

Multiplicity of eigenvalues: The *algebraic multiplicity* of a repeated eigenvalue is its times of repetition; the *geometric multiplicity* of a repeated eigenvalue is the number of its corresponding eigenvector.

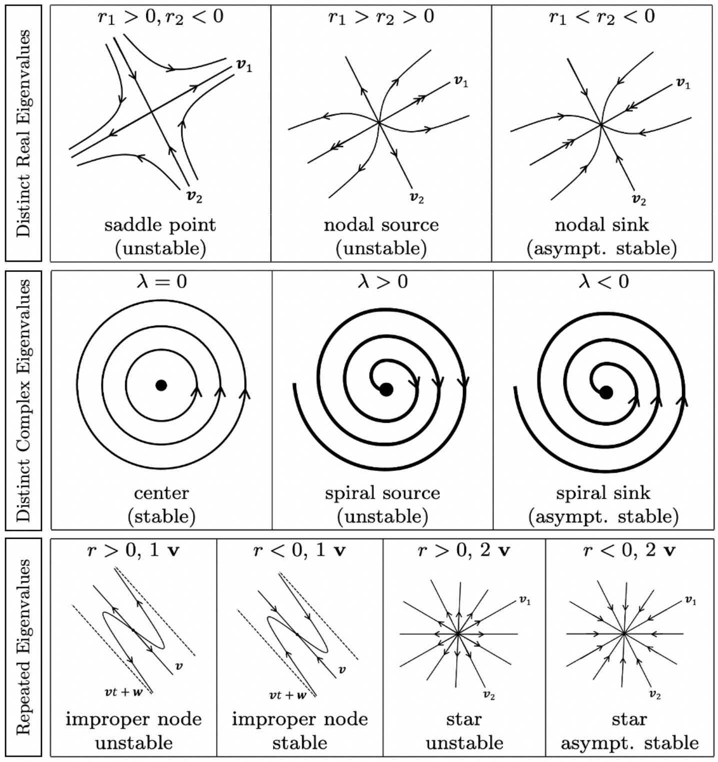
Jordan form: If $\mathbf{A}$ has an eigenvalue r with algebraic multiplicity m and geometric multiplicity p, then $\mathbf{J}$ has m number of r in its diagonal and a total of p Jordan blocks with diagonal entry r.

Solve $\mathbf{x}' = \mathbf{A}\mathbf{x}$. $\mathbf{A}$ has repeated eigenvalue.
If $\mathbf{A}$ has repeated eigenvalue r with only one eigenvector $\mathbf{v}$, then the Jordan form is:

$$J = \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}.$$

Find $\mathbf{w}$ such that $(\mathbf{A} - r\mathbf{I})\mathbf{w} = \mathbf{v}$. The general solution is:

$$\mathbf{x}(t) = C_1 e^{rt}\mathbf{v} + C_2 e^{rt}(\mathbf{v}t + \mathbf{w}).$$



Nonhomogeneous System of Equations

Solve $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t)$.
Using variation of parameters, the solution is:

$$\mathbf{x}(t) = \mathbf{\Psi}(t)\left(\int \mathbf{\Psi}^{-1}(t)\mathbf{g}(t)dt + \mathbf{C}\right).$$

Nonlinear Autonomous System

Autonomous system: $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x})$.

- Critical point (equilibrium solution): $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.
- Stability: A critical point $\mathbf{x}_0$ is *stable* if for any $\epsilon > 0$, there is a $\delta > 0$ such that for every solution satisfying the initial condition, $||\mathbf{x}(0) - \mathbf{x}_0|| < \epsilon$ for all $t > 0$; a stable critical point $\mathbf{x}_0$ is *asymptotically stable* if there exists some $\delta < 0$ such that every initial data satisfying $||\mathbf{x}(0) - \mathbf{x}_0|| < \delta$ leads to $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_0$.

Linearization: Consider $\begin{cases} x'(t) = F(x,y) \\ y'(t) = G(x,y) \end{cases}$ with equilibrium point (x_0, y_0). The corresponding linear system is $\mathbf{x}' = \mathbf{J}(\mathbf{x}_0)\mathbf{x}$:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} F_x(x_0,y_0) & F_y(x_0,y_0) \\ G_x(x_0,y_0) & G_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- For a 2×2 autonomous system $\mathbf{x}'(t) = \mathbf{f}(\mathbf{x})$ with critical point $\mathbf{x}_0$, if $\mathbf{J}(\mathbf{x}_0)$ has distinct eigenvalues and both eigenvalues have non-zero real parts, then the critical point $\mathbf{x}_0$ must have the same type and stability as in the linear system.

4 Boundary Value Problems

Eigenvalue Problem

Homogeneous boundary condition: $\begin{cases} a_1 y(\alpha) + a_2 y'(\alpha) = 0 \\ b_1 y(\beta) + b_2 y'(\beta) = 0 \end{cases}$.

- The BVP is *homogeneous* if both equation and boundary condition are homogeneous. Any homogeneous BVP has only constant zero solution $y \equiv 0$ or infinitely many solutions.

Eigenvalue problem: $Ly + \lambda y = 0$ with homogeneous BC. Here $Ly = y'' + p(x)y' + q(x)y$. We are to find all $\lambda \in \mathbb{C}$ such that the BVP has a nontrivial solution.

- **Eigenvalue:** λ such that the BVP has a nontrivial solution.
- **Eigenfunction:** Nontrivial solution y corresponding to λ.

Solve $y'' + \lambda y = 0$ with $y(0) = y(\pi) = 0$.
Characteristic equation: $r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}$.
<u>**Case I**</u> $\lambda > 0$. There are two complex roots $r = \pm\sqrt{\lambda}i$.
General solution: $y(x) = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$.
We have $\begin{cases} y(0) = C_1 = 0 \\ y(\pi) = C_2\sin(\sqrt{\lambda}\pi) = 0 \Rightarrow \sqrt{\lambda} = n \text{ for some } n \in \mathbb{N}^+. \end{cases}$
Hence $y(x) = \sin(nx)$ is the corresponding eigenfunction for $\lambda = n^2$.
<u>**Case II**</u> $\lambda = 0$. There is one repeated root $r = 0$.
General solution: $y(x) = C_1 + C_2 x$.
We have $\begin{cases} y(0) = C_1 = 0 \\ y(\pi) = C_2\pi = 0 \Rightarrow C_2 = 0 \end{cases}$ (only trivial solution).
<u>**Case III**</u> $\lambda < 0$. There is two real roots $r = \pm\sqrt{-\lambda} = \pm\sqrt{\mu}$.
General solution: $y(x) = C_1 e^{\sqrt{\mu}x} + C_2 e^{-\sqrt{\mu}x}$.
We have $\begin{cases} y(0) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2 \\ y(\pi) = C_1 e^{\sqrt{\mu}\pi} + C_2 e^{-\sqrt{\mu}x} = 0 \Rightarrow C_1(e^{\sqrt{\mu}\pi} - e^{-\sqrt{\mu}x}) = 0. \end{cases}$
Since $e^{\sqrt{\mu}\pi} - e^{-\sqrt{\mu}x} > 0$, there is only trivial solution.

Inner product: $(u,v) = \int_a^b u(x)v(x)dx$.

- u and v are orthogonal if their inner product is zero.

Prove $\lambda > 0$ without using explicit solutions.
$y'' + \lambda y = 0 \Rightarrow -y'' = \lambda y$. Assume y is nontrivial with eigenvalue λ. Here $(-y'', y) = \int_0^\pi -y''(x)y(x)dx = \int_0^\pi y'(x)^2dx - y'y\Big|_0^\pi = \int_0^\pi y'(x)^2dx - (y'(\pi)\cancel{y(\pi)} - y'(0)\cancel{y(0)})$. Hence, $\int_0^\pi y'^2dx = \lambda\int_0^\pi y^2dx$. Note that LHS ≥ 0, RHS > 0. If LHS $= 0$, then $y' \equiv 0$, hence $y \equiv 0$, which is trivial. Hence LHS > 0. $\lambda = \frac{\text{LHS}}{\text{RHS}} > 0$.

Prove that eigenfunctions corresponding to different eigenvalues must be mutually orthogonal.
Assume y_1, y_2 are nontrivial eigenfunctions corresponding to $\lambda_1 \neq \lambda_2$.
We have $\begin{cases} -y_1'' = \lambda_1 y_1 \Rightarrow (-y_1'', y_2) = \lambda_1(y_1, y_2) & ① \\ -y_2'' = \lambda_2 y_2 \Rightarrow (-y_2'', y_1) = \lambda_2(y_2, y_1) & ② \end{cases}$.
① − ②, we have $\int_0^\pi -y_1''y_2 + y_2''y_1 dx = (\lambda_1 - \lambda_2)\int_0^\pi y_1y_2dx$.
LHS $= \int_0^\pi \cancel{(y_1'y_2' - y_2'y_1')}dx - \cancel{(y_1'y_2 - y_2'y_1)\Big|_0^\pi} = 0$. Hence RHS $= 0$.
Since $\lambda_1 \neq \lambda_2$, $\int_0^\pi y_1y_2dx = 0 \Rightarrow (y_1, y_2) = 0$. They are orthogonal.

Prove that for any λ, the space of eigenfunctions is one-dimensional (simple).
Let y_1, y_2 be two eigenfunctions corresponding to λ.
$W[y_1, y_2](0) = \begin{vmatrix} \cancel{y_1(0)} & \cancel{y_2(0)} \\ y_1'(0) & y_2'(0) \end{vmatrix} = 0$. By Abel's theorem, $W \equiv 0$.
Hence y_1 and y_2 are linearly dependent. The space is one-dimensional.

Fourier sine series: Any function f on $[0,\pi]$ can be expressed as $f(x) = c_1\sin(x) + c_2\sin(2x) + ...$ Here $c_n = \frac{\int_0^\pi f(x)\sin(nx)dx}{\int_0^\pi \sin(nx)^2dx}$.

Sturm-Liouville Boundary Value Problem

Sturm-Liouville BVP: $(p(x)y')' - q(x)y + \lambda r(x)y = 0$ on $[\alpha, \beta]$ with homogeneous boundary conditions that are separated:

$$a_1 y'(\alpha) + a_2 y(\alpha) = 0$$
$$b_1 y'(\beta) + b_2 y(\beta) = 0$$

Here a_1, a_2 are not both 0, b_1, b_2 are not both 0.

- $(p(x)y')' = p(x)y'' + p'(x)y'$.
- Let $L[y] = -(p(x)y')' + q(x)y$, the equation becomes $L[y] = \lambda r(x)y$. The operator L is *self-adjoint* in the sense that $(Lu, v) = (u, Lv)$ for any u, v satisfying the BC. Here the inner product is given by $(u,v) = \int_\alpha^\beta u(x)\bar{v}(x)dx$.

Regular Sturm-Liouville BVP: If p, p', q, r are cont. on $[\alpha, \beta]$, and $p(x) > 0, r(x) > 0$ on $[\alpha, \beta]$, the SL BVP is *regular*; otherwise it is *singular*.

- All eigenvalues to regular Sturm-Liouville BVP must be real.

Prove that if $q \geq 0$ and $-py'\bar{y}\Big|_\alpha^\beta \geq 0$, then $\lambda \geq 0$.
$(L[y], y) = \int_\alpha^\beta -(p(x)y')'\bar{y}dx + \int_\alpha^\beta q(x)|y|^2dx = \int_\alpha^\beta p(x)|y'|^2dx - py'\bar{y}\Big|_\alpha^\beta + \int_\alpha^\beta q(x)|y|^2dx = (\lambda r(x)y, y) = \int_\alpha^\beta \lambda r(x)|y|^2dx$. Hence $\lambda \geq 0$.

- Eigenfunctions ϕ_m, ϕ_n from different eigenvalues $\lambda_m \neq \lambda_n$ are orthogonal with respect to the weight function $r(x)$. That is, ϕ_m, ϕ_n satisfy $\int_\alpha^\beta r(x)\phi_m(x)\phi_n(x)dx = 0$.
- There is only 1 eigenfunction for each eigenvalue (simple).
- Infinite sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$. $\lambda_n \to \infty$ as $n \to \infty$.