MA3220 Ordinary Differential Equations

AY2022/23 Semester 1 · Midterm Examination Cheatsheet · *Prepared by Tian Xiao @snoidetx*

Differential Equations

Solving a separable ODE

$$y'(t) = P(t)Q(y)$$
$$\int \frac{1}{Q(y)}\,dy = \int P(t)\,dt + C$$

Existence & uniqueness of solutions

- 1st order linear ODE: If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then for any $y_0 \in \mathbb{R}$, there exists a unique solution to the differential equation $y' + p(t)y = g(t)$ for each t in I, with initial condition $y(t_0) = y_0$.

- 1st order non-linear ODE: Consider the equation $y' = f(t, y)$ with initial condition $y(t_0) = y_0$. If f and $\frac{\partial f}{\partial y}$ are both continuous in some rectangle $R = (\alpha, \beta) \times (\gamma, \delta)$ containing the point (t_0, y_0), then in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there exists a unique solution to the IVP.

- 2nd order linear ODE: If the functions p, q, g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique solution to the differential equation $y'' + p(t)y' + q(t)y = g(t)$ for each t in I, with initial condition $y(t_0) = y_0$ and $y'(t_0) = y'_0$.

1st Order ODEs

Terminologies

- Linearity: An ODE is *linear* if it can be written in the form $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1 y = P(x)$.

- Homogeneity: $P(x) = 0$.

- Convexity: If $y''(x) > 0$, then $y(x)$ is *concave*; otherwise, it is *convex*.

- Equilibrium solution: $y'(x) = 0$.

- Exact equation: An ODE $M(x, y) + N(x, y)y' = 0$ is called an *exact ODE* if there exists a function $\psi(x, y)$ such that $\frac{\partial \psi}{\partial x}(x, y) = M(x, y)$ and $\frac{\partial \psi}{\partial y}(x, y) = N(x, y)$.

  - If an ODE is exact, $M_y = N_x$.

  - If M, N, M_y, N_x are continuous in a simply connected region $D \subset \mathbb{R}^2$, then the equation $M(x, y) + N(x, y)y' = 0$ is an exact equation if and only if $M_y = N_x$.

Solving a 1st order linear ODE

$$y' + P(x)y = Q(x)$$
$$\text{Let } \mu(x) = e^{\int P(x)dx},$$
$$\mu'(x) = \mu(x)P(x);$$
$$\mu(x)y' + \mu'(x)y = \mu(x)Q(x)$$
$$\mu(x)y = \int \mu(x)Q(x)\,dx$$
$$y = \frac{\int \mu(x)Q(x)\,dx}{\mu(x)y} + C$$

Solving a 1st order exact ODE

$$M(x, y) + N(x, y)y' = 0$$
$$\text{Run the test: Is } M_y = N_x?$$
$$\psi(x, y) = \int M(x, y)\,dx + g(y)$$
$$\text{Solve } dy \text{ by } \psi_y = N(x, y)$$
$$\text{General solution: } \psi(x, y) = C$$

Euler's method

1. Partition the interval $[x_0, X]$ into a finite number of mesh points $x_0 < x_1 < \cdots < x_n = X$. Step size $h = \frac{X - x_0}{n}$.
2. For each $i = 1, 2, \cdots, n$, $y_i = y_{i-1} + y'(i-1)h$.

2nd Order ODEs

Superposition principle

For a linear homogenous equation $L(y) = 0$, if y_1 and y_2 are solutions, then for any constant c_1 and c_2, the linear combination $c_1y_1 + c_2y_2$ is also a solution.

Wronskian and general solution

Let y_1 and y_2 be two solutions of a 2nd order linear homogenous ODE, their *Wronskian* is defined as

$$W[y_1, y_2](t) := \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y_2y'_1$$

Let y_1 and y_2 be two solutions of $y'' + p(t)y' + q(t) = 0$ in an interval I, with p, q continuous in I. Then $y(t) = c_1y_1 + c_2y_2$ is the *general solution* in I if and only if $W[y_1, y_2](t_0) \neq 0$ for some $t_0 \in I$.

Abel's theorem

Let y_1 and y_2 be two solutions of $y'' + p(t)y' + q(t) = 0$ in an interval I, with p, q continuous in I. Then their Wronskian satisfies

$$W[y_1, y_2](t) = ce^{-\int p(t)\,dt}$$

for some constant c. As a result, W is either always 0 or never 0.

Solving a 2nd order linear homogenous ODE

$$ay'' + by' + c = 0$$

Consider the solution to its *characteristic equation*

$$ar^2 + br + c = 0$$

Case I: $\Delta > 0$, $r = \lambda_1$ or λ_2.

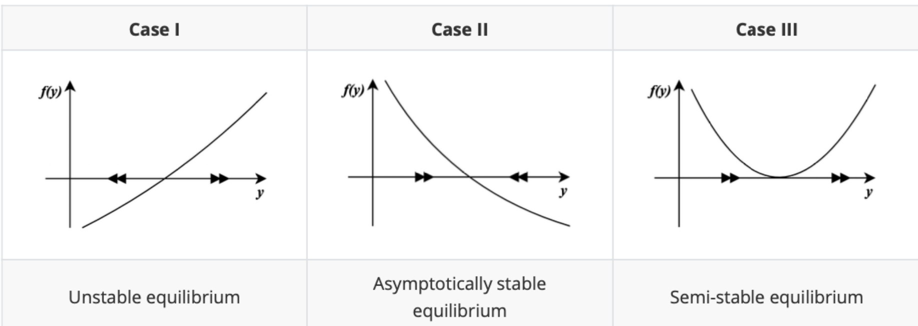
$$y = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$$

Case II: $\Delta < 0$, $r = \alpha \pm \beta i$.

$$y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

Case III: $\Delta = 0$, $r = \lambda$.

$$y = (c_1 + c_2t)e^{\lambda t}$$



Finding another solution

• Abel's theorem: Plug in the value of $y_1(t)$, $y_1'(t)$ and $ce^{-\int p(t)\,dt}$ and solve for y_2. Set $c = 1$ for convenience.

Example: Find another solution y_2 of the ODE $y'' + 4y' + 4y = 0$ given $y_1(t) = e^{-2t}$.

By Abel's Theorem,

$$\begin{vmatrix} e^{-2t} & y_2(t) \\ -2e^{-2t} & y_2'(t) \end{vmatrix} = e^{-\int 4\,dt} = e^{-4t}$$
$$e^{-2t}y_2'(t) + 2e^{-2t}y_2(t) = e^{-4t}$$
$$y_2'(t) + 2y_2(t) = e^{-2t}$$
$$y_2(t) = te^{-2t}$$

• Reduction of order: Let $y_2(t) = v(t)y_1(t)$ and plug in to the ODE.

$$y''(t) + p(t)y'(t) + q(t)y = 0$$
$$\text{Let } y_2(t) = v(t)y_1(t).$$
$$y_2' = vy_1' + v'y_1$$
$$y_2'' = vy_1'' + 2v'y_1' + v''y_1$$
$$vy_1'' + 2v'y_1' + v''y_1$$
$$+pvy_1' + pv'y_1 + qvy_1 = 0$$
$$y_1v'' + (2y_1' + py_1)v' = 0$$

Let $u = v'$ and this becomes a 1st order ODE.

2nd order linear non-homogenous ODE

• Making the right guess:

$g(t)$	guess
Ce^{kt}	Ae^{kt}
$C\sin kt$ or $C\cos kt$	$A\sin kt + B\cos kt$
degree-n polynomial	$A_nx^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0$
sum of different types of terms	sum of their respective guesses
product of different types of terms	product of their respective guesses

- We handle exceptions by multiplying t to our guess when our guess solves the corresponding homogenous equation.

• Variation of parameters: For the equation $y'' + p(t)y + q(t)y = g(t)$, let the general solution be $Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, where y_1 and y_2 are the solutions to the corresponding homogenous equation. Set $u_1'y_1 + u_2'y_2 = 0$, so that $Y' = u_1y_1' + u_2y_2'$ and $Y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$. Plug this into the ODE and we get:

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = g(t) \end{cases}$$

Solve this simultaneous equation and we get the following result:

$$\begin{cases} u_1(t) = -\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}\,dt + c_1 \\ u_2(t) = \int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}\,dt + c_2 \end{cases}$$

• Using power series: A *power series* centered at x_0 is an infinite series of the form $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$. Guess $y = f(x)$ and plug into the ODE:

$$y = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 \cdots$$
$$= \sum_{n=0}^{\infty} a_n(x - x_0)^n$$
$$y' = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 \cdots$$
$$= \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}$$
$$y'' = 2a_2 + 6a_3(x - x_0) + 12a_4(x - x_0)^2 \cdots$$
$$= \sum_{n=2}^{\infty} n(n-1)a_n(x - x_0)^{n-2}$$

Use *shift of summation index* to get a recurrence relation.

Apply the initial condition:

$$y(x_0) = a_0$$
$$y'(x_0) = a_1$$
$$\cdots$$
$$y^{(n)}(x_0) = n!a_n$$

- Convergence radius: Every power series has a *convergence radius* ρ (can be 0, positive or infinity), such that when $|x - x_0| > \rho$, the series diverges and when $|x - x_0| < \rho$, the series converges absolutely. If $f(x)$ is a polynomial, the power series of the function $\frac{1}{f(x)}$ centered at x_0 has its convergence radius equal to the distance between x_0 and the nearest complex roots of $f(x)$.

- Ratio test for convergence: Consider the expression

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| |x - x_0|$$

If the value < 1, the series converges absolutely at x; if the value $= 1$, the test is inconclusive; if the value > 1, the series diverges.

- A point t_0 is called an *ordinary point* if both $p(t)$ and $q(t)$ are analytic at t_0; otherwise it is called a *singular point*. If t_0 is an ordinary point, then the ODE has a series solution centered at t_0:

$$y(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n = a_0y_1(t) + a_1y_2(t)$$

Here y_1 and y_2 form a fundamental set of solutions, and their convergence radius is at least the minimum of the convergence radius of p and q.

Other Useful Facts

• Integration by parts:

$$\int uv'\,dx = uv - \int vu'\,dx$$

• Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

- If f has a Taylor series expansion at x_0 with a radius of convergence $\rho > 0$, then f is said to be *analytic* at x_0.

- Being analytic implies being differentiable for arbitrarily many times.

- If f and g are analytic at x_0 with a radius of convergence ρ, then fg and $f + g$ are also analytic at x_0 with a radius of convergence ρ.

• Inflection point: $y''(t) = 0$.

Good luck!