MA3252 Linear Programming

Midterm Examination Helpsheet

AY2023/24 Semester 2 · Prepared by Tian Xiao *@snoidetx*

1 Linear Programming (LP) Problem

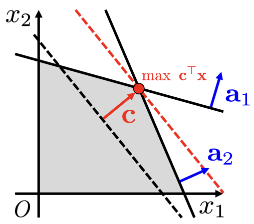
$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \text{ (or max)} \quad & \mathbf{c}^\top\mathbf{x} \\ \text{s.t.} \quad & \mathbf{a}_i^\top\mathbf{x} \geq b_i \text{ for some } i; \\ & \mathbf{a}_i^\top\mathbf{x} \leq b_i \text{ for some } i; \\ & \mathbf{a}_i^\top\mathbf{x} = b_i \text{ for some } i; \\ & x_j \geq 0 \text{ for some } j; \\ & x_j \leq 0 \text{ for some } j; \\ & x_j \in \mathbb{R} \text{ for some } j, \end{aligned}$$

where $\mathbf{a}_i = (a_{i,1}, a_{i,1}, \cdots, a_{i,n})^\top \in \mathbb{R}^n, b_i \in \mathbb{R}$.

- The feasible region $P \in \mathbb{R}^n$ is a *polyhedron*.
- An LP problem may have
  - ▷ one unique solution; OR
  - ▷ one finite optimal cost with multiple optimal solutions; OR
  - ▷ unbounded optimal cost with no optimal solution; OR
  - ▷ empty feasible set, where optimal cost equals $+\infty$.

Graphical Representation: In $\mathbb{R}^n$, $\{\mathbf{x} \mid \mathbf{a}^\top\mathbf{x} = b\}$ is a hyperplane with normal vector $\mathbf{a}$.

- Vector $\mathbf{c}$ corresponds to the direction of increasing $\mathbf{c}^\top\mathbf{x}$.



Standard Form: Minimization + equality + non-negative.

- Maximization objective: $\max \mathbf{c}^\top\mathbf{x} \Rightarrow \min -\mathbf{c}^\top\mathbf{x}$.
- Inequality constraints: $\mathbf{a}_i^\top\mathbf{x} \leq / \geq b_i \Rightarrow \begin{cases} \mathbf{a}_i^\top\mathbf{x} \pm s_i = b_i \\ s_i \geq 0 \end{cases}$.
  - ▷ s_i is *slack* variable.
- Non-positive variables: $x_i \leq 0 \Rightarrow x_i^- \geq 0$.
- Free variables: $x_i \Rightarrow (x_i^+ - x_i^-); x_i^+, x_i^- \geq 0$.

Convex Sets and Convex Functions:

- Convex set: $\forall \mathbf{x}, \mathbf{y} \in S\, \forall \lambda \in [0,1]\, [\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S]$.
- Convex combination: $\mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}^i$, where $\lambda_i \in [0,1]$ s.t. $\sum_{i=1}^{k} \lambda_i = 1$.
  - ▷ Any convex combination of two optimal solutions is also an optimal solution.
- Convex hull: Set of convex combinations.
- Convex function: $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\, \forall \lambda \in [0,1]\, [f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})]$.
  - ▷ f is *concave* if $-f$ is convex.
  - ▷ *Affine function* $d + \mathbf{c}^\top\mathbf{x}$ is both convex and concave.
  - ▷ **Thm 1.5.1.** If $f_1, f_2, \cdots, f_m : \mathbb{R}^n \to \mathbb{R}$ are convex, then $f(\mathbf{x}) = \max\{f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_m(\mathbf{x})\}$ is also convex.
    - ∗ **Cor 1.5.2.** $\max_{i=1,2,\cdots,m} \{d_i + \mathbf{c}_i^\top\mathbf{x}\}$ is convex.

Example. Reformulate as LP problem:
- $\max \min(x_1, x_2) \Rightarrow \max t$ s.t. $t \leq x_1; t \leq x_2$.
- $|x_1 - x_2| \leq 2 \Rightarrow x_1 - x_2 \leq 2; x_1 - x_2 \geq -2$.
- $\min |x| \Rightarrow \min \max(x, -x) \Rightarrow \min t$ s.t. $t \geq x; t \geq -x$.

2 Geometry of Linear Programming

Polyhedra and Extreme Points:

- Polyhedron: $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$.
  - ▷ A polyhedron is a finite intersection of half-spaces.
  - ▷ A polyhedron has finite number of vertices/BFS.
- 3 definitions of corner points: Consider a convex set $P \subseteq \mathbb{R}^n$,
  - ▷ Extreme point: A point $\mathbf{x}^* \in P$ is an *extreme point* if whenever points $\mathbf{y}, \mathbf{z} \in P$ and scalar $\lambda \in (0,1)$ are such that $\mathbf{x}^* = \lambda\mathbf{y} + (1-\lambda)\mathbf{z}$, we have $\mathbf{y} = \mathbf{z} = \mathbf{x}^*$.
  - ▷ Vertex: A point $\mathbf{x}^* \in P$ is a *vertex* if there is a $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c}^\top\mathbf{x}^* > \mathbf{c}^\top\mathbf{y}$ for all $\mathbf{y} \in P \setminus \{\mathbf{x}^*\}$.
  - ▷ Basic feasible solution (BFS): $\mathbf{x}^*$ is a *BFS* of a polyhedron if n **linearly independent** constraints are active at $\mathbf{x}^*$ and $\mathbf{x}^* \in P$.
    - ∗ Basic solution: A point where n linearly independent constraints are active but not necessarily in P.
  - ▷ **Thm 2.1.5.** In a non-empty polyhedron, an extreme point, a vertex and a BFS are equivalent.
- Degenerate: A basic solution (not necessarily feasible) is *degenerate* if more than n contraints are active at $\mathbf{x}^*$.

Basic Feasible Solutions for Standard Polyhedra:

$$\{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\},$$

where $\mathbf{A} \in \mathbf{R}^{m\times n}, m < n$ contains m linearly independent rows.

- Basic solution for standard polyhedra: $\mathbf{x}^*$ is a *basic solution* iff
  - ▷ the equality constraints $\mathbf{A}\mathbf{x}^* = \mathbf{b}$ hold; AND
  - ▷ $x_i^* = 0$ for $n - m$ indices; AND
  - ▷ these n binding constraints are linearly independent.
- **Thm 2.2.1.** A vector $\mathbf{x}^* \in \mathbb{R}^n$ is a basic solution of the standard form LP iff
  - ▷ $\mathbf{A}\mathbf{x}^* = \mathbf{b}$; AND
  - ▷ There exists $B = \{B(1), B(2), \cdots, B(m)\} \subset \{1, 2, \cdots, n\}$ such that
    - ∗ the columns of $\mathbf{A}_B = \left(\mathbf{A}_{B(1)}, \mathbf{A}_{B(2)}, \cdots, \mathbf{A}_{B(m)}\right)$ are linearly independent; AND
    - ∗ $x_i^* = 0$ for $i \in N = \{1, 2, \cdots, n\} \setminus B$.

  If in addition, $\mathbf{x}_B^* \geq \mathbf{0}$, then $\mathbf{x}^*$ is a BFS.
  - ▷ $\mathbf{x}_B^* = \mathbf{A}_B^{-1}\mathbf{b}$.
  - ▷ A degenerate basic solution $\mathbf{x}^*$ has more than $n - m$ zero components.
  - ▷ If $n = m + 1$, then there are at most two BFSs.
- Adjacent BFS: Extreme points connected by an edge on the boundary.
  - ▷ The corresponding bases share all but one basic column.
  - ▷ There are common $n - 1$ linearly independent constraints that are active at both of them.

Optimal Solutions at Extreme Points:

- A polyhedron $P \subseteq \mathbb{R}^n$ *contains a line* if $\exists \mathbf{x}^* \in P\, \exists \mathbf{d} \neq \mathbf{0} \in \mathbb{R}^n\, \forall \lambda \in \mathbb{R}\, [\mathbf{x}^* + \lambda\mathbf{d} \in P]$. A polyhedron containing an infinite line does not contain an extreme point.
- **Thm 2.3.1.** Let $\mathbf{A} \subseteq \mathbb{R}^{m\times n}, m \geq n$. Suppose $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}\} \neq \emptyset$. The following are equivalent:
  - ▷ P does not contain a line;
  - ▷ P has a BFS;
  - ▷ P has n linearly independent constraints.
  - ▷ Implication: Every non-empty bounded polyhedron and every non-empty standard form polyhedron has at least one BFS.
- **Thm 2.3.3.** If an LP has a BFS and an optimal solution, then there exists an optimal solution that is a BFS.
  - ▷ Hence, it suffices to check BFS.

3 The Simplex Method

Feasible Direction and Reduced Cost:

- Feasible direction: For a polyhedron P and a point $\mathbf{x} \in P$, a vector d is a *feasible direction* if $\mathbf{x} + \theta\mathbf{d} \in P$ for some $\theta > 0$.
  - ▷ For standard polyhedra, $\mathbf{A}\mathbf{d} = \mathbf{0}$.
- **Clm †.** Let $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$ with $\mathbf{x}_B \geq 0, \mathbf{x}_N = 0$ be a BFS. A direction d moving from $\mathbf{x}$ to an adjacent BFS is of the form $\mathbf{d}^j = \left(\mathbf{d}_B^j, \mathbf{d}_N^j\right)$ for some $j \in N$, where
  - ▷ $\mathbf{d}_N^j = \mathbf{e}_j$ where $e_{j,j} = 1$ and $e_{j,i} = 0$ for $i \in N \setminus \{j\}$; AND
  - ▷ $\mathbf{d}_B^j = -\mathbf{A}_B^{-1}\mathbf{A}_j$.
- Reduced cost: Let $\mathbf{x}$ be a basic solution. Let $\mathbf{c} = (\mathbf{c}_B, \mathbf{c}_N)$. For each $j \in \{1, 2, \cdots, n\}$, the *reduced cost* $\bar{c}_j$ of variable x_j is defined by
  $$\bar{c}_j = c_j - \mathbf{c}_B^\top \mathbf{A}_B^{-1}\mathbf{A}_j.$$
  - ▷ For $j \in B$, $\bar{c}_j = 0$.
  - ▷ If $\bar{c}_j \geq 0$ for all $j \in N$, then current BFS is the unique optimal solution.
  - ▷ A direction $\mathbf{d}^j$ is an *improving direction* if $\bar{c}_j < 0$.
  - ▷ Change in cost in any direction $\mathbf{d}$:
    $$\mathbf{c}^\top\mathbf{d} = \mathbf{c}_B^\top\mathbf{d}_B + \mathbf{c}_N^\top\mathbf{d}_N = -\mathbf{c}_B^\top\mathbf{A}_B^{-1}\mathbf{A}_N\mathbf{d}_N + \mathbf{c}_N^\top\mathbf{d}_N.$$
- **Clm.** Let $\mathbf{x}$ be a BFS with basis B. Any feasible direction at $\mathbf{x}$ can be represented as
  $$\sum_{j\in N} \lambda_j \mathbf{d}^j \text{ for } \lambda_j \geq 0.$$
- Degenerate: A BFS is *degenerate* if some element of $\mathbf{x}_B$ is zero. A BFS is *non-degenerate* if $\mathbf{x}_B = \mathbf{A}_B^{-1}\mathbf{b} > \mathbf{0}$.

- **Thm 3.1.6.** (Optimality conditions) Consider a BFS $\mathbf{x}$ associated with basis matrix $\mathbf{A}_B$, and let $\bar{\mathbf{c}}$ be corresponding vector of reduced costs.
  - ▷ If $\bar{\mathbf{c}} \geq \mathbf{0}$, then $\mathbf{x}$ is optimal.
  - ▷ If $\mathbf{x}$ is optimal and non-degenerate, then $\bar{\mathbf{c}} \geq \mathbf{0}$.

Simplex Method:

① Start with basis B and its basic columns $\mathbf{A}_B$ and BFS $\mathbf{x}$.
  - ▷ Check that $\mathbf{x}$ is indeed a BFS.

② Compute reduced costs $\bar{c}_j = c_j - \mathbf{c}_B^\top \mathbf{A}_B^{-1} \mathbf{A}_j$ for all $j \in N$.
  - ▷ If $\bar{c}_j \geq 0$ for all $j \in N$, then current BFS is optimal. END.
  - ▷ Otherwise, choose some j for which $\bar{c}_j < 0$.

③ Compute $\mathbf{d}_B^j = -\mathbf{A}_B^{-1}\mathbf{A}_j$ (see **Clm †.**).
  - ▷ If $\mathbf{d}_B^j \geq \mathbf{0}$, then problem is unbounded. END.
  - ▷ Otherwise, let $\theta^* = \min\left\{ \frac{x_i}{-d_i^j} \,\middle|\, i \in B, d_i < 0 \right\}$.

④ Let $l \in B$ be such that $\theta^* = \frac{x_l}{-d_l^j}$. The corresponding x_l is the leaving variable.

⑤ Form a new basis $\bar{B} = (B \setminus \{l\}) \cup \{j\}$.

⑥ The other basic variables are $x_i + \theta^* d_i^j$ for $i \neq l$.

⑦ The entering variable x_j assumes $\theta^* = \frac{x_l}{-d_l^j}$. Go to Step ①.

Big-M Method:

① Multiply constraints by -1 to make $\mathbf{b} \geq \mathbf{0}$ as needed.

② Add artificial variables $y_1, y_2, \cdots, y_m$ to constraints without positive slack.

③ Apply simplex method on LP with cost $\min \mathbf{c}^\top \mathbf{x} + M \sum_{y=1}^{m} y_i$, where $M \gg 0$ is treated as some algebraic variable.

Tableau Method:

① Start from basis B and its basic columns $\mathbf{A}_B$ (preferably $\mathbf{I}$, and the corresponding BFS $\mathbf{x} = (\mathbf{x}_B, \mathbf{x}_N)$ (check)).

Basic	$x_j, j \in N$	$x_{B(1)}$	$x_{B(2)}$	Solution
$\mathbf{c}$	c_j	$c_{B(1)}$	$c_{B(2)}$	
$\bar{\mathbf{c}}$	$c_j - \mathbf{c}_B^\top \mathbf{A}_B^{-1} \mathbf{A}_j$	0	0	Obj: $-\mathbf{c}_B^\top \mathbf{x}_B$
$B(1)$	$-d_1^j = \left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_1$	1	0	$x_{B(1)}$
$B(2)$	$-d_2^j = \left(\mathbf{A}_B^{-1}\mathbf{A}_j\right)_2$	0	1	$x_{B(2)}$

② Choose some j such that $j < 0$. At that column, for all $-d_i^j > 0, i \in B$, calculate $\frac{x_i}{-d_i^j}$ and pick the smallest one i^* (0 is also considered).

③ i^* leaves and j enters. Normalize the row where this happens such that the cell $(x_j, x_j) = 1$.

④ Perform row operations to all rows including $\bar{\mathbf{c}}$ such that the column of x_j is all 0 but one 1.

⑤ If all $\bar{\mathbf{c}} \geq \mathbf{0}$, END; else, return to ② again.

Two-Phase Method:

Phase I: Find BFS using auxiliary LP.

① Multiply constraints by -1 to make $\mathbf{b} \geq \mathbf{0}$ as needed.

② Add artificial variables $y_1, y_2, \cdots, y_m$ to constraints without positive slack.

③ Apply simplex method on auxiliary LP with cost $\min \sum_{y=1}^{m} y_i$.

④ If the optimal cost in auxiliary LP is:
  - ▷ zero: A BFS to original LP is found.
  - ▷ positive: Original LP is infeasible. END.

Phase II: Solve original LP.

① Take BFS found in Phase I to start Phase II.

② Use cost coefficients of original LP to compute reduced costs.

③ Apply simplex method to original LP.
  - ▷ Either finds an optimum, or detects unboundedness.