MA3264 Mathematical Modelling
AY2022/23 Sem I

1^{st}-order Models

BLACK HOLE

Black holes lose their mass

$$\frac{dM}{dt} = -\frac{hc^4}{15360\pi G^2 M^2}$$

- Life span: $T = \frac{5120\pi G^2 M^3}{hc^4}$

HEAT FLOW

$$\frac{dT}{dt} = k(T - T_0)$$

FALL WITH AIR RESISTANCE

From $F = ma$, we have

$$m\frac{dv}{dt} = mg - bv^2$$

$$v = k\coth\left(\frac{bkt}{m} - c\right)$$

ORBIT OF PLANET

$$\left(\frac{du}{d\theta}\right)^2 + (u - A)^2 = B^2$$

where $u(\theta) = \frac{1}{r(\theta)}$, $\frac{B}{A} < 1$

$$r = \frac{A^{-1}}{1 + \frac{B}{A}\sin(\theta + C)}$$

VIRGA

$$\begin{cases} V = aA^{\frac{3}{2}} \\ \frac{dV}{dt} = -bA \end{cases}$$

Time for complete evaporation $= \frac{3aA_0^{\frac{1}{2}}}{b}$

MOTH

$$r = Re^{\frac{\theta}{\tan(\psi)}}$$

"spiral" '飞蛾扑火'

SALT DISSOLVE IN WATER

$$\frac{dQ}{dt} = \underbrace{3 \times 0.25}_{\text{inflow}} - \underbrace{\frac{3Q}{100}}_{\text{overflow}}$$

☆ $\lim Q = 25$

Uranium-Thorium DATING ☆☆☆☆☆

$$\begin{cases} \frac{dU}{dt} = -k_U U \\ \frac{dT}{dt} = +k_U U - k_T T \end{cases}$$

Getting k_U from half-life: $k_U = \frac{\ln 2}{\text{half-life}}$

$$\begin{cases} U = U_0 e^{-k_U t} \\ T = \frac{k_U}{k_T - k_U} U_0 (e^{-k_U t} - e^{-k_T t}) \end{cases}$$

The ratio

$$\frac{T}{U} = \frac{k_U}{k_T - k_U}[1 - e^{(k_U - k_T)t}]$$

Each ratio corresponds to a time.

POTATO BLIGHT

$$\frac{dx}{dt} = K(1 - x(t))(x(t-p) - x(t-p-q))$$

"latency" sick but not infectious infectious period

$$x(t) = 1 - (1 - \alpha)e^{-K\int_0^t f(\tau)d\tau}$$

α: initial sick proportion

Final sick proportion:

$$\beta = 1 - (1 - \alpha)e^{-Kq\beta}$$

THE ARCTIC OCEAN

$$\begin{cases} \frac{dI}{dt} = -aI \\ \frac{dW}{dt} = bW \end{cases}$$

I(ice) W(water) constant

with seasonal variation:

$$\begin{cases} \frac{dI}{dt} = -I \\ \frac{dW}{dt} = \sin(t)W \end{cases}$$

$I + W$ constant.

CABLE CATENARY

$$\frac{dy}{dx} = \frac{\mu}{T}\int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\, dt$$

μ: weight per unit len; T: horizontal component of tension

$$y = \frac{T}{\mu}\cosh\left(\frac{\mu}{T}x\right) - \frac{T}{\mu}$$

PERFORMANCE CURVE

$$\frac{dP}{dt} = C[M - P] \Rightarrow P = M - Me^{-Ct}$$

(M: max)

If $C = k\tanh\left(\frac{t}{T}\right) \Rightarrow P = M(1 - \operatorname{sech}^{kT}\left(\frac{t}{T}\right))$

(learning ability ↑)

SPREAD OF RUMOUR

$$\frac{dR}{dt} = kR(45 - R) \Rightarrow \frac{1}{R} = \frac{1}{45} + Ce^{-45kt}$$

2^{nd}-order Models

EARTH STOPS ROTATING

Fact: $\frac{d^2y}{dx^2} = \frac{dy}{dx}\frac{d}{dy}y' = \frac{d}{dy}\left(\frac{y'^2}{2}\right)$ (chain rule)

$$\frac{d}{dr}\left(\frac{\dot{r}^2}{2}\right) = -\frac{GM}{r^2}$$

$$t = \frac{R^{\frac{3}{2}}}{\sqrt{2GM}}\int_{\frac{2}{3}}^{1}\frac{dx}{\sqrt{\frac{1}{x} - 1}}$$

(time taken to reach Venus)

Graph of Hyperbolic Functions

$y = \sinh x = -i\sin(ix) = \frac{e^x - e^{-x}}{2}$

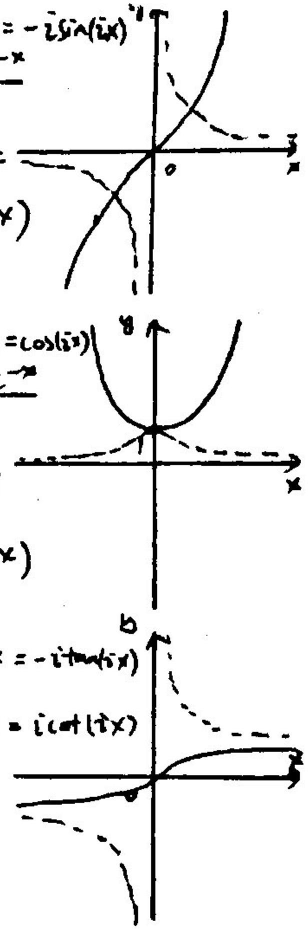
$y = \operatorname{csch} x = i\csc(ix)$

$y = \cosh x = \cos(ix) = \frac{e^x + e^{-x}}{2}$

$y = \operatorname{sech} x = \sec(ix)$

$y = \tanh x = -i\tan(ix)$

$y = \coth x = i\cot(ix)$



All the Best!!

University Integration and Differential Equation

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Trigonometric Identities

- sin, cos: $\sin^2 x + \cos^2 x = 1$
- tan: $\tan x = \frac{\sin x}{\cos x}$
- sec, csc: $\sec x = \frac{1}{\cos x}$; $\csc x = \frac{1}{\sin x}$;
- cot: $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$
- $\sec^2 x - \tan^2 x = 1$; $\csc^2 x - \cot^2 x = 1$
- $\sin(x+y) = \sin x \cos y + \sin y \cos x$
- $\sin 2x = 2 \sin x \cos x$
- $\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = \cos^2 x - 1$
- $\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
- $\tan \frac{x}{2} = \pm\sqrt{(1-\cos x)(1+\cos x)}$
- $\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\sin x \sin y = \frac{\cos(x+y) - \cos(x-y)}{2}$
- $\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2}$
- $\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}$
- sinh, cosh: $\cosh^2 x - \sinh^2 x = 1$
  $\sinh x = \frac{e^x - e^{-x}}{2}$; $\cosh x = \frac{e^x + e^{-x}}{2}$
- tanh: $\tanh x = \frac{\sinh x}{\cosh x}$
- sech, csch: $\operatorname{sech} x = \frac{1}{\cosh x}$; $\operatorname{csch} x = \frac{1}{\sinh x}$
- coth: $\coth x = \frac{1}{\tanh x}$
- $\tanh^2 x + \operatorname{sech}^2 x = 1$; $\coth^2 x - \operatorname{csch}^2 x = 1$
- $\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$
- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
- $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

Common Integrals

Basic

- $\int k \, dx = kx + C$
- $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$
- $\int \frac{1}{x} \, dx = \ln|x| + C$
- $\int e^x \, dx = e^x + C$

Fractional

- $\int \frac{1}{ax+b} = \frac{1}{a} \ln|ax+b| + C$
- $\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{a^2+x^2}} \, dx = \sinh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{x^2-a^2}} \, dx = \cosh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{a^2-x^2} \, dx = \frac{1}{a} \tanh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{x\sqrt{1-x^2}} \, dx = -\operatorname{sech}^{-1} x + C$
- $\int \frac{1}{|x|\sqrt{1+x^2}} \, dx = -\operatorname{csch}^{-1} x + C$

Logarithmic

- $\int \ln x \, dx = x \ln x - x + C$

Trigonometric

- $\int \cos x \, dx = \sin x + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \tan x \, dx = \ln|\sec x| + C$
- $\int \sec x \, dx = \ln|\sec u + \tan u| + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \csc x \cot x \, dx = -\csc x + C$
- $\int \csc^2 x \, dx = -\cot x + C$
- $\int \sinh x \, dx = \cosh x + C$
- $\int \cosh x \, dx = \sinh x + C$
- $\int \operatorname{sech}^2 x \, dx = \tanh x + C$
- $\int \operatorname{csch}^2 x \, dx = -\coth x + C$
- $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$
- $\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$

Special Integrals

- Partial fractions
- Integration by parts:
  $\int u \, dv = uv - \int v \, du$
- $\int \sin^n x \cos^m x \, dx$:
  Use trigonometric identities to convert it into $\sin^k x \cos x$ or $\cos^k x \sin x$.

Differential Equations

1. $M(x) - N(y)y' = 0$

(*Separable*) Separate the variables x and y and rewrite the equation as $\int M(x) \, dx = \int N(y) \, dy$.

2. $y' + P(x)y = Q(x)$

Multiply both sides by an **integrating factor** $\mu(x) = e^{\int P(x) \, dx}$:

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$

$$\mu(x)y = \int \mu(x)Q(x) \, dx$$

3. $y' + P(x)y = Q(x)y^n$

(*Bernoulli*) Let $z = y^{1-n}$, then $z' = (1-n)y^{-n}y'$. Hence we have:

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

$$\frac{z'}{1-n} + P(x)z = Q(x)$$

and use integrating factor.

4. $ay'' + by' + cy = 0$

Consider the **characteristic equation** $ax^2 + bx + c = 0$ with roots λ_1 and λ_2:

- If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.
- If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then $y = (c_1 + c_2 x)e^{\lambda x}$.
- If $\lambda_1 \neq \lambda_2 = \alpha \pm \beta i \in \mathbb{C}$, then $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.

5. $ay'' + by' + cy = r(x), r(x) \neq 0$

The goal is to find the **particular solution** y_p:

- If $r(x)$ is a polynomial of order n, guess $y_p(x)$ to be a n-th order polynomial.
- If $r(x)$ is in the form of $g(x)e^{kx}$, let $y_p(x) = u(x)e^{kx}$.
- If $r(x)$ is in the form of $g(x)\cos kx$ or $u(x)\sin kx$, let $z(x) = u(x)e^{ikx}$ and take $Re(z)$ or $Im(z)$).

Other Useful Formulae

- (*Fundamental Theorem of Calculus*) $\frac{d}{dx}\int_a^x f(t) \, dt = f(x)$
- (*Binomial Expansion*) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$