MA4207 Mathematical Logic

AY2022/23 Semester 2 · Prepared by Tian Xiao *@snoidetx*

Sentential Logic

Well-Formed Formulas (WFFs): $(\neg \mathbf{A}_i)$ or $(\mathbf{A}_i \to \mathbf{A}_j)$.
Length of Formulas: Every symbol, atom or bracket has length 1.
Structural Induction for Formulas If a set of formulaes $\mathbf{S}$ contains all atoms and truth-constants and is closed under $\mathcal{E}_\neg, \mathcal{E}_\to, \mathcal{E}_\leftrightarrow, \mathcal{E}_\wedge, \mathcal{E}_\vee, \mathcal{E}_\oplus$ then $\mathbf{S}$ contains every WFF.
Proposition Let $\mathbf{S}$ be the set of all expressions which have the same amount of opening and closing brackets. Then $\mathbf{S}$ contains all WFFs.
Truth Assignment: A truth-assignment ν is a mapping which assigns to every atom a truth-value ($\mathbf{0}$ or $\mathbf{1}$). $\bar{\nu}$ extends ν to all WFFs:

1. $\bar{\nu}(\mathbf{0}) = \mathbf{0}; \bar{\nu}(\mathbf{1}) = \mathbf{1}; \bar{\nu}(\mathbf{A_k}) = \nu(\mathbf{A_k})$;
2. $\bar{\nu}(\neg\alpha) = 1 - \alpha$;
3. $\bar{\nu}(\alpha \wedge \beta) = \bar{\nu}(\alpha) \cdot \bar{\nu}(\beta)$;
4. $\bar{\nu}(\alpha \vee \beta) = \bar{\nu}(\alpha) + \bar{\nu}(\beta) - \bar{\nu}(\alpha) \cdot \bar{\nu}(\beta)$;
5. $\bar{\nu}(\alpha \to \beta) = \bar{\nu}((\neg\alpha) \vee \beta)$;
6. $\bar{\nu}(\alpha \oplus \beta) = \bar{\nu}(\alpha) + \bar{\nu}(\beta) - \mathbf{2} \cdot \bar{\nu}(\alpha) \cdot \bar{\nu}(\beta)$;
7. $\bar{\nu}(\alpha \leftrightarrow \beta) = \bar{\nu}(\neg(\alpha \oplus \beta))$.

Tautological Implication: If $\mathbf{X}$ is a set of WFFs and α is a formula, one says that $\mathbf{X}$ *tautologically implies* α (written $\mathbf{X} \vDash \alpha$) iff every truth-assignment ν for atoms occurring in $\mathbf{X}$ or α satisfies that whenever $\bar{\nu}(\beta) = \mathbf{1}$ for all $\beta \in \mathbf{X}$ then $\bar{\nu}(\alpha) = 1$. α is a *tautology* iff $\emptyset \vDash \alpha$.
Satisfiable: A set of formulaes $\mathbf{X}$ is *satisfiable* iff there is a truth-assignment ν such that $\bar{\nu}(\alpha) = \mathbf{1}$ for all $\alpha \in \mathbf{X}$.
Compactness Theorem If $\mathbf{X}$ is an infinite set of formulaes such that every finite subset $\mathbf{Y} \subseteq \mathbf{X}$ is satisfiable, then $\mathbf{X}$ itself is also satisfiable.
Lemma 13A Every WFF has the same number of opening and closing brackets.
Lemma 13B If a WFF is split into two non-empty expressions α and β, then α has more opening than closing brackets and β has more closing than opening brackets.
Polish Notation: $\alpha \wedge \beta$ becomes $\wedge\alpha\beta$.
Rules for Omitting Brackets

1. The outmost bracket can be omitted.
2. $\neg$ binds to what follows it directly.
3. $\wedge$ binds more than $\vee$, than $\to$, than $\leftrightarrow$.
4. $\alpha \to \beta \to \gamma$ is bracketed as $\alpha \to (\beta \to \gamma)$.

Subformulas: Given a WFF $\alpha = \mathbf{a_1 a_2} \cdots \mathbf{a_n}$, β is a subformula of α iff β is a WFF and furthermore $\beta = \mathbf{a_i a_{i+1}} \cdots \mathbf{a_j}$ for some i, j with $1 \le i \le j \le n$.
Proposition Every construction sequence for a WFF α contains besides α all subformulas of α.
Recursion Theorem Assume that $\mathbf{C}$ is freely generated subset of $\mathbf{D}$ with respect to a base set $\mathbf{B}$ and constructor functions f, g and further assume that the "generation is free", that is, for each $\mathbf{x} \in \mathbf{C}$, exactly one of the following cases holds:

1. $\mathbf{x} \in \mathbf{B}$;
2. There are $\mathbf{y}, \mathbf{z} \in \mathbf{C}$ with $\mathbf{x} = \mathbf{f}(\mathbf{y}, \mathbf{z})$ and $\mathbf{y}, \mathbf{z}$ uniquely depend on $\mathbf{x}$;
3. There is an $\mathbf{y} \in \mathbf{C}$ with $\mathbf{x} = \mathbf{g}(\mathbf{y})$ and $\mathbf{y}$ uniquely depends on $\mathbf{x}$.

Furthermore, assume that there is a further set $\mathbf{E}$ and that there are functions $\mathbf{h} : \mathbf{B} \to \mathbf{E}$, $\bar{\mathbf{f}} : \mathbf{E} \times \mathbf{E} \to \mathbf{E}$, $\bar{\mathbf{g}} : \mathbf{E} \to \mathbf{E}$. Then there is a unique function $\bar{\mathbf{h}}$ such that

1. For all $\mathbf{x} \in \mathbf{B}$, $\bar{\mathbf{h}}(\mathbf{x}) = \mathbf{h}(\mathbf{x})$;
2. For all $\mathbf{x}, \mathbf{y} \in \mathbf{C}$, $\bar{\mathbf{h}}(\mathbf{f}(\mathbf{x}, \mathbf{y})) = \bar{\mathbf{f}}(\bar{\mathbf{h}}(\mathbf{x}), \bar{\mathbf{h}}(\mathbf{y}))$;
3. For all $\mathbf{x} \in \mathbf{C}$, $\bar{\mathbf{h}}(\mathbf{g}(\mathbf{x})) = \bar{\mathbf{g}}(\bar{\mathbf{h}}(\mathbf{x}))$.

Majority Connective: $\#(\alpha, \beta, \gamma)$.
Boolean Function: For a WFF α using atoms $\mathbf{A_1}, \cdots, \mathbf{A_n}$, one can define the Boolean function $\mathbf{B}^\mathbf{n}_\alpha(\mathbf{x_1}, \cdots, \mathbf{x_n})$.
Partial Order on Boolean Functions: $\mathbf{B}^\mathbf{n}_\alpha \le \mathbf{B}^\mathbf{n}_\beta$ iff for all $\{\mathbf{x_1}, \cdots, \mathbf{x_n}\} \in \{\mathbf{0}, \mathbf{1}\}^\mathbf{n}$, $\mathbf{B}^\mathbf{n}_\alpha(\mathbf{x_1}, \cdots, \mathbf{x_n}) \le \mathbf{B}^\mathbf{n}_\beta(\mathbf{x_1}, \cdots, \mathbf{x_n})$.
Theorem 15A Let α and β be WFFs whose sentence symbols are among $\mathbf{A_1}, \cdots, \mathbf{A_n}$. The following statements are true:

1. $\{\alpha\} \vDash \beta$ iff $\mathbf{B}^\mathbf{n}_\alpha \le \mathbf{B}^\mathbf{n}_\beta$;
2. $\{\alpha\} \vDash \beta$ and $\{\beta\} \vDash \alpha$ iff $\mathbf{B}^\mathbf{n}_\alpha = \mathbf{B}^\mathbf{n}_\beta$;
3. $\emptyset \vDash \alpha$ iff $\mathbf{B}^\mathbf{n}_\alpha = \mathbf{B}^\mathbf{n}_\mathbf{1}$.

Theorem 15B For every $\mathbf{n}$-place Boolean function $\mathbf{f}$ there is a WFF α using atoms $\mathbf{A_1}, \cdots, \mathbf{A_n}$ such that $\mathbf{f} = \mathbf{B}^\mathbf{n}_\alpha$ (disjunction of conjunctive clauses).
Disjunctive Normal Form: Disjunction of conjunctive clauses.
Conjunctive Normal Form: Conjunction of disjunctive clauses.
Corollary 15C For every α one can find an equivalent β in disjunctive normal form.
Completeness: A set $\mathbf{S}$ of connectives is *complete* if all Boolean functions with at least one input variable can be represented by a formula α using these connectives.
Theorem 15D The following sets of connectives are complete: $\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \vee\}$. $\{\wedge, \vee\}$ and $\{\oplus, \wedge\}$ are not complete.
Fuzzy Logic:

1. $\mathbf{0} = \min(\mathbf{Q})$;
2. $\mathbf{1} = \max(\mathbf{Q})$;
3. If $\mathbf{p}, \mathbf{q} \in \mathbf{Q}$ then $\mathbf{1} - \mathbf{p}, \min\{\mathbf{p} + \mathbf{q}, \mathbf{2} - \mathbf{p} - \mathbf{q}\} \in \mathbf{Q}$.

Fuzzy Connectives:

1. $p \wedge q = \min\{p, q\}$;
2. $p \vee q = \max\{p, q\}$;
3. $\neg p = 1 - p$;
4. $p \to q = \min\{1 + q - p, 1\}$;
5. $p \leftrightarrow q = \min\{1 + q - p, 1 + p - q\}$;
6. $p \oplus q = \min\{p + q, 2 - p - q\}$.

Fuzzy Tautologies:

1. $\mathbf{S} \vDash \alpha$ iff for every ν there is a $\beta \in \mathbf{S} \cup \{\mathbf{1}\}$ with $\bar{\nu}(\beta) \le \bar{\nu}(\alpha)$.
2. α is a tautology iff $\emptyset \vDash \alpha$.
3. All formulas made from atoms and connectives $\wedge$, $\vee$, $\neg$ are evaluated to $\frac{\mathbf{1}}{\mathbf{2}}$ in the case that $\frac{\mathbf{1}}{\mathbf{2}} \in \mathbf{Q}$ and all atoms take the value $\frac{\mathbf{1}}{\mathbf{2}}$ and hence none of them is a tautology.
4. $\alpha \leftrightarrow \alpha$ and $\alpha \to \alpha$ are tautologies and indeed, $\alpha \leftrightarrow \beta$ is a tautology iff for all $\mathbf{Q}$-valued truth-assignments ν, $\bar{\nu}(\alpha) = \bar{\nu}(\beta)$.

Corollary 17A If $\mathbf{S} \vDash \alpha$ then there is a finite $\mathbf{S}' \subseteq \mathbf{S}$ with $\mathbf{S}' \vDash \alpha$.

- Works also for fuzzy logic where $\mathbf{Q}$ is a finite set.

Compactness Theorem for Fuzzy Logic Let $\mathbf{Q}$ be a compact set of possible truth-values for fuzzy logic (i.e. $\mathbf{Q} = \left\{\mathbf{0}, \frac{\mathbf{1}}{\mathbf{k}}, \cdots, \frac{\mathbf{k-1}}{\mathbf{k}}, \mathbf{1}\right\}$ or $\mathbf{Q} = \{\mathbf{r} \in \mathbb{R} : \mathbf{0} \le \mathbf{r} \le \mathbf{1}\}$). Then a countable set $\mathbf{S}$ of formulas is satisfiable iff $\mathbf{S}$ is finitely satisfiable.
Notions of Effectiveness:

1. There is a program in a usual programming language (e.g. JavaScript) which computes the function.
2. One defines the function from some basic functions by recursion in one variables where + and − and comparison-functions are defined. Furthermore, if $\mathbf{f}$ is a recursive function, then one can define a new function $\mathbf{g}$ which maps $\mathbf{x}, \mathbf{y}$ to the least $\mathbf{z}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{0}$; note that $\mathbf{g}(\mathbf{x}, \mathbf{y})$ is undefined if such a $\mathbf{z}$ cannot be found.
3. An effective function $\mathbf{f}$ is given by a graph which is a Diophantine set, that is, there is a polynomial $\mathbf{p}$ with coefficients from $\mathbb{Z}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ iff there are $\mathbf{z_1}, \cdots, \mathbf{z_k}$ with $\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z_1}, \cdots, \mathbf{z_k}) = \mathbf{0}$. While the coefficients of the polynomial are integers which can be negative, the values for $\mathbf{x}, \mathbf{y}, \mathbf{z_1}, \cdots, \mathbf{z_k}$ are all from $\mathbb{N}$.

Decidable Sets: A set $\mathbf{L}$ is *decidable* iff there is a recursive function $\mathbf{f}$ such that for all inputs $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) = \mathbf{1}$ when $\mathbf{x} \in \mathbf{L}$ and $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ when $\mathbf{x} \notin \mathbf{L}$. That is, some algorithm can determine which possible inputs are in the set $\mathbf{L}$ and which are not.
Recursively Enumerable Sets: A set $\mathbf{L}$ is *recursively enumerable* iff there is an algorithm (i.e. recursive function) which enumerates the members of $\mathbf{L}$; $\emptyset$ is defined to be recursively enumerable.
Theorem 17B There is an algorithm which can check whether an expression α is a well-formed formula; that is, the set of all WFFs is decidable.
Theorem 17C Given a finite set of well-formed formulas $\mathbf{S}$ and a well-formed formula α, one can decide whether $\mathbf{S} \vDash \alpha$.
Corollary 17D If $\mathbf{S}$ is finite then the set $\{\alpha : \alpha$ is a WFF and $\mathbf{S} \vDash \alpha\}$ is decidable.
Theorem 17E A set is recursively enumerable iff there is an algorithm which, for all $\mathbf{x} \in \mathbf{A}$, outputs "yes"; however, for $\mathbf{x} \notin \mathbf{A}$, the function $\mathbf{f}(\mathbf{x})$ might either never output something or output "no".
Theorem 17F (Kleene's Theorem) A set $\mathbf{A}$ is decidable iff both the set $\mathbf{A}$ and its complement are recursively enumerable.
Theorem 17G If $\mathbf{S}$ is a recursively enumerable set of formulas then $\{\alpha : \alpha$ is a WFF and $\mathbf{S} \vDash \alpha\}$ is recursively enumerable.
Theorem 17H If $\mathbf{S}$ is a recursively enumerable set of formulas then there is a further decidable set $\mathbf{T}$ of formulas with $\forall$ WFF α $[\mathbf{S} \vDash \alpha$ iff $\mathbf{T} \vDash \alpha]$.
Closure Properties: The following statements are true for recursively enumerable and decidable subsets of $\mathbb{N}$:

1. Every infinite recursively enumerable set $\mathbf{X}$ has an infinite recursive subset $\mathbf{Y}$.
2. If $\mathbf{X}$ and $\mathbf{Y}$ are both recursively enumerable, so are $\mathbf{X} \cup \mathbf{Y}$ and $\mathbf{X} \cap \mathbf{Y}$.
3. If $\mathbf{X}$ and $\mathbf{Y}$ are both decidable, so are $\mathbf{X} \cup \mathbf{Y}$, $\mathbf{X} \cap \mathbf{Y}$ and $\mathbb{N} - \mathbf{X}$.

First-Order Logic

Atomic Formulas: $P(t_1, t_2, \cdots, t_n)$ for an n-ary predicate and n terms; OR $t_1 = t_2$; OR q for a truth-value q.
Well-Formed Formulas (WFFs): Atomic formulas + Logical connectives + Quantifiers ($\exists x\, [P(x)]$ is equivalent to $\neg\forall v_i\, [\neg P(v_i)]$).
Free Occurence: Occurence of variable in atomic formulas is free; occurence of v_i within the range of $\forall v_i\, [\alpha]$ is bound.
Sentence: Every WFF which satisfies $h(\alpha) = \emptyset$ is a sentence.
Structure: A structure $\mathfrak{A}$ consists of a non-empty domain A which assigns to every constant, function and predicate a value from A or from functions of $A^n \to A$ or from predicates of $A^n \to Q$.

Theorem 22A Given a structure $\mathfrak{A}$, a formula α and two default assignments to the variables s_1 and s_2. If this agrees on all variables which occur free in α, then $\mathfrak{A}, s_1 \vDash \alpha$ iff $\mathfrak{A}, s_2 \vDash \alpha$.
Corollary 22B If α is a sentence, then either $\mathfrak{A}, s \vDash \alpha$ for all s or for no s.
Logical Implication: Let S be a set of WFFs and α be a WFF. Then $S \vDash \alpha$ iff for every structure $\mathfrak{A}$ and every value-assignment s to the variables, if $\mathfrak{A}, s \vDash \beta$ for all $\beta \in S$ then $\mathfrak{A}, s \vDash \alpha$.
Logical Equivalence: Two formulas are logically equivalent iff $\{\alpha\} \vDash \beta$ and $\{\beta\} \vDash \alpha$.
Validity: A formula α is *valid* iff $\emptyset \vDash \alpha$.
Satisfiability: A formula α is *satisfiable* iff there is a structure $\mathfrak{A}$ and a value-assignment s such that $\mathfrak{A}, s \vDash \alpha$.
Corollary 22C $S \vDash \alpha$ depends only on the structures involved and not on the s considered.
Definability in a Structure: An n-ary relation R/n-ary function f/constant c is *definable* in a structure $\mathfrak{A}$ iff there is a first-order formula α with free variables $v_1, v_2, \cdots, v_n$ such that for all $a_1, a_2, \cdots, a_n \in A$,

$$R(a_1, a_2, \cdots, a_n) \text{ is true} \Leftrightarrow \mathfrak{A}, s(v_1|a_1, v_2|a_2, \cdots, v_n|a_n) \vDash \alpha;$$
$$f(a_1, a_2, \cdots, a_n) = b \Leftrightarrow \mathfrak{A}, s(v_1|a_1, v_2|a_2, \cdots, v_n|a_n, v_{n+1}|b) \vDash \alpha;$$
$$a = c \Leftrightarrow \mathfrak{A}, s(v_1|a) \vDash \alpha.$$

Definability of Classes of Structures: A class of structures is *definable* iff there is a set S of axioms which defines it.
Elementary Class: A class of structures is called *elementary* iff there is a single sentence α such that a structure belongs to C iff it satisfies the formula α.

- Elementary in a wider sense: Replace α with S.

Homomorphism: A mapping h from the domain A of $\mathfrak{A}$ to the domain B of $\mathfrak{B}$ is called a *homomorphism* iff it satisfies the following conditions:

1. For every constant symbol c, $h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.
2. For every n-ary function symbol f and all $a_1, a_2, \cdots, a_n \in A$, $h\left(f^{\mathfrak{A}}(a_1, \cdots, a_n)\right) = f^{\mathfrak{B}}(h(a_1), \cdots, h(a_n))$.
3. For every n-ary predicate symbol P and all $a_1, a_2, \cdots, a_n \in A$, $P^{\mathfrak{A}}(a_1, \cdots, a_n) \Rightarrow P^{\mathfrak{B}}(h(a_1), \cdots, h(a_n))$.

Strong Homomorphism: $P^{\mathfrak{A}}(a_1, \cdots, a_n) \Leftrightarrow P^{\mathfrak{B}}(h(a_1), \cdots, h(a_n))$.
Isomorphism: Strong homomorphism + injective + surjective.
Homomorphism Theorems

1. $\forall t\, [h(\bar{s}(t)) = \bar{s}'(t)]$.
2. For every atomic formula α, if $\mathfrak{A}, s \vDash \alpha$ then $\mathfrak{B}, s' \vDash \alpha$.
3. If h is a strong homomorphism and α has no equality or quantifier, $\mathfrak{A}, s \vDash \alpha$ iff $\mathfrak{B}, s' \vDash \alpha$.
4. If h is a strong injective homomorphism and α has no quantifer, $\mathfrak{A}, s \vDash \alpha$ iff $\mathfrak{B}, s' \vDash \alpha$.
5. If h is a strong surjective homomorphism and α has no equality, $\mathfrak{A}, s \vDash \alpha$ iff $\mathfrak{B}, s' \vDash \alpha$.
6. If h is an isomorphism, then $\mathfrak{A}, s \vDash \alpha$ iff $\mathfrak{B}, s' \vDash \alpha$.

Elementary Equivalence: Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* iff they use the same logical language and for all sentences α, $\mathfrak{A} \vDash \alpha$ iff $\mathfrak{B} \vDash \alpha$.
Corollary 22D Isomorphic structures are all elementarily equivalent.
Axioms for Formal Proofs

- **Modus Ponens**: If $S \vDash \alpha$ and $S \vDash \alpha \to \beta$, then $S \vDash \beta$.
- **Axiom 1**: α for every α which is obtained by taking a tautology in sentential logic and replacing all axioms by WFFs in a consistent way (the same atom needs always be replaced by the same formula).
- **Axiom 2**: $\forall x[\alpha] \to (\alpha)^x_t$ for all well-formed formula α, variables x and terms t where α does not have any variable names inside t which are bound at the place x.
- **Axiom 3**: $\forall x\, [\alpha \to \beta] \to \forall x\, [\alpha] \to \forall x\, [\beta]$.
- **Axiom 4**: $\alpha \to \forall x\, [\alpha]$ if x does not occur free in α.
- **Axiom 5**: $x = x$ for every variable x.
- **Axiom 6**: $x = y \to \alpha \to \beta$ for all variables x, y and all atomic formulas α and all β derived from α by replacing some occurences of x by occurences of y or vice versa.
- **Axiom 7**: $\forall x\, [\alpha]$ whenever α is already in Λ and x is any variable.
- **Deduction Theorem**: If $S \cup \{\alpha\} \vdash \beta$, then $S \vdash \alpha \to \beta$.
- **Contraposition**: $S \cup \{\alpha\} \vdash \beta$, then $S \cup \{\neg\beta\} \vdash \neg\alpha$.
- **Generalisation Theorem**: If $S \vdash \alpha$ and the variable x does not occur free in S, then $S \vdash \forall x\, [\alpha]$.
- **Corollary 24G**: Assume that $S \vdash (\alpha)^x_c$, where the constant symbol c neither occurs in S nor in α, then $S \vdash \forall x\, [\alpha]$.
- **Existential Instantiation**: Assume c does not occur free in S. If $S \cup \{(\alpha)^x_c\} \vdash \alpha$ then $S \cup \{\exists x\, [\alpha]\} \vdash \alpha$.

Tautological Implication: If n formulas $\beta_1, \cdots, \beta_n$ tautologically imply α, then $\{\beta_1, \cdots, \beta_n\} \vdash \alpha$.
Inconsistency: A set S of formulas is *inconsistent* if one can derive an antitautology, so that every formula can be derived from S.
Reductio and Absurdum: If $S \cup \{\alpha\}$ is inconsistent, then $S \vdash \neg\alpha$.
Soundness: A proof-system is called *sound* iff it only proves correct theorems, that is, whenever $S \vdash \alpha$ we have $S \vDash \alpha$.
Completeness: A proof-system is called *complete* iff it proves every correct theorem, that is, whenever $S \vDash \alpha$ we have $S \vdash \alpha$.
Lemma If a formula α is valid, then so is $\forall x\, [\alpha]$.
Soundness Theorem If $S \vdash \alpha$, then $S \vDash \alpha$.
Gödel's Completeness Theorem If $S \vdash \alpha$, then $S \vDash \alpha$.
Compactness Theorem Let S be a set of WFFs and α be a WFF. Then the following statements hold:

1. If $S \vDash \alpha$ then S has a finite subset T such that $T \vDash \alpha$.
2. If every finite subset of S is satisfiable, so is S.

Reasonable Language: A logical language is reasonable iff it has at most countably many logical symbols and an algorithm can enumerate these symbols together with information where they are constants, functions or predicates and their arities.
Enumerability Theorem Let the logical language be reasonable and T be a recursively enumerable set of WFFs. Then the set $\{\alpha : T \vDash \alpha\}$ is recursively enumerable.
Corollary If S is a recursive enumerable set of formulas such that for each α, either $S \vDash \alpha$ or $S \vDash \neg\alpha$ but not both, then the set of formulas logically implied by S is decidable.
Theory: A *theory* is a set T of sentences with the property that for all sentences α, if $T \vdash \alpha$ then $\alpha \in T$. The theory of a structure $\mathfrak{A}$ is the set of all sentences which are true in A.
Theorem 26A If a set S of sentences has arbitrarily large finite models, then S has an infinite model.
Corollary 26B Consider some fixed logical language. The class of all infinite structures in this language is not an elementary class, but it is an elementary class in a wider sense. The class of all finite structures in this language is not an elementary class in a wider sense.
Theorem 26C Assume that a finite structure $\mathfrak{A}$ has a finite language. Then the theory of $\mathfrak{A}$ is decidable.
Corollary 26D Assume that a logical language is reasonable, then one can recursively enumerate the set T of all sentences α such that there is a finite model $\mathfrak{A}$ satisfying α.
Theorem 26E Let the logical language be finite.

1. Assume that S is a recursively enumerable set of sentences. Now one can enumerate all finite structures $\mathfrak{A}$ not satisfied by S; this is done by forming for each finite structure $\mathfrak{A}$ the sentence $\alpha_{\mathfrak{A}}$ and then enumerate $\mathfrak{A}$ whenever one has found a proof for $S \vdash \neg\alpha_{\mathfrak{A}}$.
2. Given S as above, one can enumerate all sentences β, such that $S \cup \{\beta\}$ does not satisfy all finite models such that there is a finite model $\mathfrak{A}$ with $S \cup \{\beta\} \vdash \neg\alpha_{\mathfrak{A}}$.

Trakhtenbrot's Theorem The set T of all sentences α which are true in all finite structures is, for most logical languages, neither decidable nor recursively enumerable.
Theorem of Lövenheim and Skolem Let the logical language be at most countable and S be the set of all WFFs. Then

1. If S is satisfiable then S has an at most countable model.
2. If S has a model then S has an at most countable model.

Axiomatisable: A theory T is *axiomatisable* iff there is a decidable set S of sentences such that a sentence α belongs to T iff $S \vdash \alpha$. A theory T is *finitely axiomatisable* iff the S witnessing that T is axiomatisable can be chosen ot be finite.
Theorem 26H If S is a set of sentences and $T = \{\alpha : S \vdash \alpha\}$ is finitely axiomatisable, then there is a finite subset S' of S such that $T = \{\alpha : S' \vdash \alpha\}$.
Corollary 26I The following are equivalent:

- T is axiomatisable;
- T is recursively enumerable;
- There is a recursively enumerable subset $S \subseteq T$ with $T = \{\alpha : S \vdash \alpha\}$;
- There is a recursively enumerable subset $S \subseteq T$ with $T = \{\alpha : S \vDash \alpha\}$.

κ-Categorical: A theory T is *κ-categorical* iff (a) it has a model of cardinality κ and (b) any two models of cardinality κ are isomorphic.
Los-Vaught Test: If a theory is κ-categorical for some κ and for each α_n either $\alpha_n \in T$ or $\neg\alpha_n \in T$ then T is complete.
Theorem 26J The theory of algebraically closed fields of characteristic 0 is complete and decidable.
Theorem 26K The theory of the dense linear orders without endpoints is $\aleph_0$-categorical and thus decidable. However, this theory is not $\aleph_1$-categorical.