MA4207 Mathematical Logic

AY2022/23 Semester 2 · Prepared by Tian Xiao *@snoidetx*

Sentential Logic

Well-Formed Formulas (WFFs): $(\neg \mathbf{A}_i)$ or $(\mathbf{A}_i \to \mathbf{A}_j)$.
Length of Formulas: Every symbol, atom or bracket has length 1.
Structural Induction for Formulas If a set of formulaes $\mathbf{S}$ contains all atoms and truth-constants and is closed under $\mathcal{E}_\neg, \mathcal{E}_\to, \mathcal{E}_\leftrightarrow, \mathcal{E}_\wedge, \mathcal{E}_\vee, \mathcal{E}_\oplus$ then $\mathbf{S}$ contains every WFF.
Proposition Let $\mathbf{S}$ be the set of all expressions which have the same amount of opening and closing brackets. Then $\mathbf{S}$ contains all WFFs.
Truth Assignment: A truth-assignment ν is a mapping which assigns to every atom a truth-value ($\mathbf{0}$ or $\mathbf{1}$). $\bar{\nu}$ extends ν to all WFFs:

1. $\bar{\nu}(\mathbf{0}) = \mathbf{0}; \bar{\nu}(\mathbf{1}) = \mathbf{1}; \bar{\nu}(\mathbf{A_k}) = \nu(\mathbf{A_k})$;
2. $\bar{\nu}(\neg\alpha) = 1 - \alpha$;
3. $\bar{\nu}(\alpha \wedge \beta) = \bar{\nu}(\alpha) \cdot \bar{\nu}(\beta)$;
4. $\bar{\nu}(\alpha \vee \beta) = \bar{\nu}(\alpha) + \bar{\nu}(\beta) - \bar{\nu}(\alpha) \cdot \bar{\nu}(\beta)$;
5. $\bar{\nu}(\alpha \to \beta) = \bar{\nu}((\neg\alpha) \vee \beta)$;
6. $\bar{\nu}(\alpha \oplus \beta) = \bar{\nu}(\alpha) + \bar{\nu}(\beta) - \mathbf{2} \cdot \bar{\nu}(\alpha) \cdot \bar{\nu}(\beta)$;
7. $\bar{\nu}(\alpha \leftrightarrow \beta) = \bar{\nu}(\neg(\alpha \oplus \beta))$.

Tautological Implication: If $\mathbf{X}$ is a set of WFFs and α is a formula, one says that $\mathbf{X}$ *tautologically implies* α (written $\mathbf{X} \vDash \alpha$) iff every truth-assignment ν for atoms occurring in $\mathbf{X}$ or α satisfies that whenever $\bar{\nu}(\beta) = \mathbf{1}$ for all $\beta \in \mathbf{X}$ then $\bar{\nu}(\alpha) = 1$. α is a *tautology* iff $\emptyset \vDash \alpha$.
Satisfiable: A set of formulaes $\mathbf{X}$ is *satisfiable* iff there is a truth-assignment ν such that $\bar{\nu}(\alpha) = \mathbf{1}$ for all $\alpha \in \mathbf{X}$.
Compactness Theorem If $\mathbf{X}$ is an infinite set of formulaes such that every finite subset $\mathbf{Y} \subseteq \mathbf{X}$ is satisfiable, then $\mathbf{X}$ itself is also satisfiable.
Lemma 13A Every WFF has the same number of opening and closing brackets.
Lemma 13B If a WFF is split into two non-empty expressions α and β, then α has more opening than closing brackets and β has more closing than opening brackets.
Polish Notation: $\alpha \wedge \beta$ becomes $\wedge\alpha\beta$.
Rules for Omitting Brackets

1. The outmost bracket can be omitted.
2. $\neg$ binds to what follows it directly.
3. $\wedge$ binds more than $\vee$, than $\to$, than $\leftrightarrow$.
4. $\alpha \to \beta \to \gamma$ is bracketed as $\alpha \to (\beta \to \gamma)$.

Subformulas: Given a WFF $\alpha = \mathbf{a_1 a_2} \cdots \mathbf{a_n}$, β is a subformula of α iff β is a WFF and furthermore $\beta = \mathbf{a_i a_{i+1}} \cdots \mathbf{a_j}$ for some i, j with $1 \le i \le j \le n$.
Proposition Every construction sequence for a WFF α contains besides α all subformulas of α.
Recursion Theorem Assume that $\mathbf{C}$ is freely generated subset of $\mathbf{D}$ with respect to a base set $\mathbf{B}$ and constructor functions f, g and further assume that the "generation is free", that is, for each $\mathbf{x} \in \mathbf{C}$, exactly one of the following cases holds:

1. $\mathbf{x} \in \mathbf{B}$;
2. There are $\mathbf{y}, \mathbf{z} \in \mathbf{C}$ with $\mathbf{x} = \mathbf{f}(\mathbf{y}, \mathbf{z})$ and $\mathbf{y}, \mathbf{z}$ uniquely depend on $\mathbf{x}$;
3. There is an $\mathbf{y} \in \mathbf{C}$ with $\mathbf{x} = \mathbf{g}(\mathbf{y})$ and $\mathbf{y}$ uniquely depends on $\mathbf{x}$.

Furthermore, assume that there is a further set $\mathbf{E}$ and that there are functions $\mathbf{h} : \mathbf{B} \to \mathbf{E}$, $\bar{\mathbf{f}} : \mathbf{E} \times \mathbf{E} \to \mathbf{E}$, $\bar{\mathbf{g}} : \mathbf{E} \to \mathbf{E}$. Then there is a unique function $\bar{\mathbf{h}}$ such that

1. For all $\mathbf{x} \in \mathbf{B}$, $\bar{\mathbf{h}}(\mathbf{x}) = \mathbf{h}(\mathbf{x})$;
2. For all $\mathbf{x}, \mathbf{y} \in \mathbf{C}$, $\bar{\mathbf{h}}(\mathbf{f}(\mathbf{x}, \mathbf{y})) = \bar{\mathbf{f}}(\bar{\mathbf{h}}(\mathbf{x}), \bar{\mathbf{h}}(\mathbf{y}))$;
3. For all $\mathbf{x} \in \mathbf{C}$, $\bar{\mathbf{h}}(\mathbf{g}(\mathbf{x})) = \bar{\mathbf{g}}(\bar{\mathbf{h}}(\mathbf{x}))$.

Majority Connective: $\#(\alpha, \beta, \gamma)$.
Boolean Function: For a WFF α using atoms $\mathbf{A_1}, \cdots, \mathbf{A_n}$, one can define the Boolean function $\mathbf{B}^\mathbf{n}_\alpha(\mathbf{x_1}, \cdots, \mathbf{x_n})$.
Partial Order on Boolean Functions: $\mathbf{B}^\mathbf{n}_\alpha \le \mathbf{B}^\mathbf{n}_\beta$ iff for all $\{\mathbf{x_1}, \cdots, \mathbf{x_n}\} \in \{\mathbf{0}, \mathbf{1}\}^\mathbf{n}$, $\mathbf{B}^\mathbf{n}_\alpha(\mathbf{x_1}, \cdots, \mathbf{x_n}) \le \mathbf{B}^\mathbf{n}_\beta(\mathbf{x_1}, \cdots, \mathbf{x_n})$.
Theorem 15A Let α and β be WFFs whose sentence symbols are among $\mathbf{A_1}, \cdots, \mathbf{A_n}$. The following statements are true:

1. $\{\alpha\} \vDash \beta$ iff $\mathbf{B}^\mathbf{n}_\alpha \le \mathbf{B}^\mathbf{n}_\beta$;
2. $\{\alpha\} \vDash \beta$ and $\{\beta\} \vDash \alpha$ iff $\mathbf{B}^\mathbf{n}_\alpha = \mathbf{B}^\mathbf{n}_\beta$;
3. $\emptyset \vDash \alpha$ iff $\mathbf{B}^\mathbf{n}_\alpha = \mathbf{B}^\mathbf{n}_\mathbf{1}$.

Theorem 15B For every $\mathbf{n}$-place Boolean function $\mathbf{f}$ there is a WFF α using atoms $\mathbf{A_1}, \cdots, \mathbf{A_n}$ such that $\mathbf{f} = \mathbf{B}^\mathbf{n}_\alpha$ (disjunction of conjunctive clauses).
Disjunctive Normal Form: Disjunction of conjunctive clauses.
Conjunctive Normal Form: Conjunction of disjunctive clauses.
Corollary 15C For every α one can find an equivalent β in disjunctive normal form.
Completeness: A set $\mathbf{S}$ of connectives is *complete* if all Boolean functions with at least one input variable can be represented by a formula α using these connectives.
Theorem 15D The following sets of connectives are complete: $\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \vee\}$. $\{\wedge, \vee\}$ and $\{\oplus, \wedge\}$ are not complete.
Fuzzy Logic:

1. $\mathbf{0} = \min(\mathbf{Q})$;
2. $\mathbf{1} = \max(\mathbf{Q})$;
3. If $\mathbf{p}, \mathbf{q} \in \mathbf{Q}$ then $\mathbf{1} - \mathbf{p}, \min\{\mathbf{p} + \mathbf{q}, \mathbf{2} - \mathbf{p} - \mathbf{q}\} \in \mathbf{Q}$.

Fuzzy Connectives:

1. $p \wedge q = \min\{p, q\}$;
2. $p \vee q = \max\{p, q\}$;
3. $\neg p = 1 - p$;
4. $p \to q = \min\{1 + q - p, 1\}$;
5. $p \leftrightarrow q = \min\{1 + q - p, 1 + p - q\}$;
6. $p \oplus q = \min\{p + q, 2 - p - q\}$.

Fuzzy Tautologies:

1. $\mathbf{S} \vDash \alpha$ iff for every ν there is a $\beta \in \mathbf{S} \cup \{\mathbf{1}\}$ with $\bar{\nu}(\beta) \le \bar{\nu}(\alpha)$.
2. α is a tautology iff $\emptyset \vDash \alpha$.
3. All formulas made from atoms and connectives $\wedge$, $\vee$, $\neg$ are evaluated to $\mathbf{\frac{1}{2}}$ in the case that $\mathbf{\frac{1}{2}} \in \mathbf{Q}$ and all atoms take the value $\mathbf{\frac{1}{2}}$ and hence none of them is a tautology.
4. $\alpha \leftrightarrow \alpha$ and $\alpha \to \alpha$ are tautologies and indeed, $\alpha \leftrightarrow \beta$ is a tautology iff for all $\mathbf{Q}$-valued truth-assignments ν, $\bar{\nu}(\alpha) = \bar{\nu}(\beta)$.

Corollary 17A If $\mathbf{S} \vDash \alpha$ then there is a finite $\mathbf{S}' \subseteq \mathbf{S}$ with $\mathbf{S}' \vDash \alpha$.

- Works also for fuzzy logic where $\mathbf{Q}$ is a finite set.

Compactness Theorem for Fuzzy Logic Let $\mathbf{Q}$ be a compact set of possible truth-values for fuzzy logic (i.e. $\mathbf{Q} = \left\{\mathbf{0}, \mathbf{\frac{1}{k}}, \cdots, \mathbf{\frac{k-1}{k}}, \mathbf{1}\right\}$ or $\mathbf{Q} = \{\mathbf{r} \in \mathbb{R} : \mathbf{0} \le \mathbf{r} \le \mathbf{1}\}$). Then a countable set $\mathbf{S}$ of formulas is satisfiable iff $\mathbf{S}$ is finitely satisfiable.
Notions of Effectiveness:

1. There is a program in a usual programming language (e.g. JavaScript) which computes the function.
2. One defines the function from some basic functions by recursion in one variables where $+$ and $-$ and comparison-functions are defined. Furthermore, if $\mathbf{f}$ is a recursive function, then one can define a new function $\mathbf{g}$ which maps $\mathbf{x}, \mathbf{y}$ to the least $\mathbf{z}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{0}$; note that $\mathbf{g}(\mathbf{x}, \mathbf{y})$ is undefined if such a $\mathbf{z}$ cannot be found.
3. An effective function $\mathbf{f}$ is given by a graph which is a Diophantine set, that is, there is a polynomial $\mathbf{p}$ with coefficients from $\mathbb{Z}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ iff there are $\mathbf{z_1}, \cdots, \mathbf{z_k}$ with $\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z_1}, \cdots, \mathbf{z_k}) = \mathbf{0}$. While the coefficients of the polynomial are integers which can be negative, the values for $\mathbf{x}, \mathbf{y}, \mathbf{z_1}, \cdots, \mathbf{z_k}$ are all from $\mathbb{N}$.

Decidable Sets: A set $\mathbf{L}$ is *decidable* iff there is a recursive function $\mathbf{f}$ such that for all inputs $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) = \mathbf{1}$ when $\mathbf{x} \in \mathbf{L}$ and $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ when $\mathbf{x} \notin \mathbf{L}$. That is, some algorithm can determine which possible inputs are in the set $\mathbf{L}$ and which are not.
Recursively Enumerable Sets: A set $\mathbf{L}$ is *recursively enumerable* iff there is an algorithm (i.e. recursive function) which enumerates the members of $\mathbf{L}$; $\emptyset$ is defined to be recursively enumerable.
Theorem 17B There is an algorithm which can check whether an expression α is a well-formed formula; that is, the set of all WFFs is decidable.
Theorem 17C Given a finite set of well-formed formulas $\mathbf{S}$ and a well-formed formula α, one can decide whether $\mathbf{S} \vDash \alpha$.
Corollary 17D If $\mathbf{S}$ is finite then the set $\{\alpha : \alpha \text{ is a WFF and } \mathbf{S} \vDash \alpha\}$ is decidable.
Theorem 17E A set is recursively enumerable iff there is an algorithm which, for all $\mathbf{x} \in \mathbf{A}$, outputs "yes"; however, for $\mathbf{x} \notin \mathbf{A}$, the function $\mathbf{f}(\mathbf{x})$ might either never output something or output "no".
Theorem 17F (Kleene's Theorem) A set $\mathbf{A}$ is decidable iff both the set $\mathbf{A}$ and its complement are recursively enumerable.
Theorem 17G If $\mathbf{S}$ is a recursively enumerable set of formulas then $\{\alpha : \alpha \text{ is a WFF and } \mathbf{S} \vDash \alpha\}$ is recursively enumerable.
Theorem 17H If $\mathbf{S}$ is a recursively enumerable set of formulas then there is a further decidable set $\mathbf{T}$ of formulas with $\forall$ WFF α $[\mathbf{S} \vDash \alpha$ iff $\mathbf{T} \vDash \alpha]$.
Closure Properties: The following statements are true for recursively enumerable and decidable subsets of $\mathbb{N}$:

1. Every infinite recursively enumerable set $\mathbf{X}$ has an infinite recursive subset $\mathbf{Y}$.
2. If $\mathbf{X}$ and $\mathbf{Y}$ are both recursively enumerable, so are $\mathbf{X} \cup \mathbf{Y}$ and $\mathbf{X} \cap \mathbf{Y}$.
3. If $\mathbf{X}$ and $\mathbf{Y}$ are both decidable, so are $\mathbf{X} \cup \mathbf{Y}$, $\mathbf{X} \cap \mathbf{Y}$ and $\mathbb{N} - \mathbf{X}$.