MA4229 Fourier Analysis and Approximation

AY2022/23 Semester 1 · *Prepared by Tian Xiao @snoidetx*

1 Fourier Series

Integration by Parts

[1] When f is piecewise continuous on $[a,b]$, then
$$\int_a^b f(x)\,\mathrm{d}x = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} g_j(x)\,\mathrm{d}x.$$
[2] When P is a polynomial of degree $< m$ and f is continuous,
$$\int Pf\,\mathrm{d}x = PF_1 - P'F_2 + P''F_3 - \cdots + \cdots + (-1)^m P^{(m)} F_{m+1} + C.$$
Here F_n refers to the n-th antiderivative of f.

[3] When P is a polynomial or other nice function and f is only piecewise continuous, then we will do [1] before [2].

Piecewise Continuous: A function f is said to be piecewise continuous on $[a,b]$ if it has a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that f is uniformly continuous on each interval (x_{i-1}, x_i) for $i = 1, \cdots, n$.

- $f(a^+)$, $f(b^-)$, $f(x_i^-)$, $f(x_i^+)$ exist for all $i = 1, \cdots, n-1$.
- Piecewise Smooth: A function f is said to be piecewise smooth on $[a,b]$ if both f' and f are piecewise continuous on $[a,b]$.

Fourier Cosine series: Assume f is piecewise continuous, then

$$[0,\pi]: \quad f(x) = a_0 + \sum_{k=1}^\infty a_k \cos(kx) \quad \begin{cases} a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,\mathrm{d}x \\ a_k = \frac{2}{\pi}\int_0^\pi f(x)\cos(kx)\,\mathrm{d}x \end{cases}$$

$$[0,L]: \quad f(x) = a_0 + \sum_{k=1}^\infty a_k \cos\left(\tfrac{k\pi x}{L}\right) \begin{cases} a_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x \\ a_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)\,\mathrm{d}x \end{cases}$$

Fourier Sine series: Assume f is piecewise continuous, then

$$[0,\pi]: \quad f(x) = \sum_{k=1}^\infty b_k \sin(kx) \;\text{———}\; b_k = \tfrac{2}{\pi}\int_0^\pi f(x)\sin(kx)\,\mathrm{d}x$$

$$[0,L]: \quad f(x) = \sum_{k=1}^\infty b_k \sin\left(\tfrac{k\pi x}{L}\right) \;\text{———}\; b_k = \tfrac{2}{L}\int_0^L f(x)\sin\left(\tfrac{k\pi x}{L}\right)\,\mathrm{d}x$$

Fourier series: Assume f is piecewise continuous with period 2π or L:

$$2\pi: \quad f(x) = a_0 + \sum_{k=1}^\infty a_k\cos(kx) + \sum_{k=1}^\infty b_k \sin(kx) \quad \begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\ a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,\mathrm{d}x \\ b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,\mathrm{d}x \end{cases}$$

$$L: \quad f(x) = a_0 + \sum_{k=1}^\infty a_k\cos\left(\frac{2k\pi x}{L}\right) + \sum_{k=1}^\infty b_k \sin\left(\frac{2k\pi x}{L}\right) \quad \begin{cases} a_0 = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\,\mathrm{d}x \\ a_k = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2k\pi x}{L}\right)\,\mathrm{d}x \\ b_k = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\left(\frac{2k\pi x}{L}\right)\,\mathrm{d}x \end{cases}$$

2 Fourier Analysis

Pointwise Convergence: Assume f is a piecewise smooth function on $[0,L]$. Then its Fourier series $[a_0 + \sum a_k \cos + \sum b_k \sin]$ converges to $\frac{f(x^+)+f(x^-)}{2}$ for all $x \in (0,L)$. *[same for Cosine and Sine series]*

- When $x = 0$ or L, the series converges to $\frac{f(0^+)+f(L^-)}{2}$.
- If continuous at $x_0 \in (0,L)$, the Fourier series converge to $f(x_0)$.
- Corollary 4.4 Assume f is a piecewise continuous function on $[0,L]$ OR $\int_0^L |f(x)|\,\mathrm{d}x < \infty$, then its Fourier coefficients $\lim_{k\to\infty} a_k = 0$ and $\lim_{k\to\infty} b_k = 0$. *[same for Cosine and Sine series]*

Uniform & Absolute Convergence: Let $\mathcal{S}_{2\pi}$ be the space of infinitely differentiable functions of period 2π. For any function $f \in \mathcal{S}_{2\pi}$, its Fourier series converges uniformly and absolutely to f.

- Proposition 4.12 If $b_k \searrow 0$, the Fourier sine series $\sum b_k \sin$ converges uniformly on $[\delta, \pi - \delta]$ for all $0 < \delta < \frac{\pi}{2}$.

Differentiability of Fourier Series: Let f be a continuous function of period 2π such that its derivative f' is piecewise continuous on $[-\pi,\pi]$. Then the Fourier series of f, $[a_0 + \sum a_k \cos + \sum b_k \sin]$, is differentiable at each point $x_0 \in (-\pi,\pi)$ at which the second derivative f'' exists:
$$f'(x_0) = \sum_{k=1}^\infty k(-a_k\sin(kx_0) + b_k\cos(kx_0)).$$

- Theorem 4.13 If f is a continuous function of period 2π such that f' is piecewise continuous on $[-\pi,\pi]$, then $ka_k, kb_k \to 0$ as $k \to \infty$.
  - ▷ If $f \in \mathcal{S}_{2\pi}$, then $k^n a_k, k^n b_k \to 0$ as $k \to \infty$ for all $n \in \mathbb{N}$.

Fourier Series of Complex-Valued Functions:
$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx} \text{ where } c_k = \hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,\mathrm{d}x.$$

- The family $\langle e^{ikx} : k \in \mathbb{Z}\rangle$ is orthogonal when $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x$.
- If f is piecewise continuous on $[0,2\pi]$, then $\sum_{k=0}^\infty \hat{f}(k)z^k$ converges on the open unit disk $\{|z|<1\}$ and hence analytic on this disk.
- Define $\tilde{f}(z) = \sum_{k\in\mathbb{Z}} \hat{f}(k)z^k$, then $\tilde{f}(e^{ix}) = f(x)$ if f is piecewise smooth, continuous and of period 2π.

Convolution: Let f and g be both periodic (of period 2π) piecewise continuous functions on $[-\pi,\pi]$. Then we define its convolution as
$$f * g(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)dy.$$

- ① $f*g = g*f$; ② $(f*g)*h = f*(g*h)$; ③ $(\alpha f_1 + f_2)*g = \alpha(f_1*g) + f_2*g$; ④ $f*g$ is continuous.
- $\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$.
- Dirichlet Kernel: $D_n(t) = \frac{1}{2} + \sum_{k=1}^n \cos kt = \frac{\sin\left(\frac{2n+1}{2}t\right)}{2\sin\left(\frac{t}{2}\right)}, t \neq 2k\pi$.
  - ▷ $\int_0^\pi D_n(t)dt = \frac{\pi}{2}$.
  - ▷ $(f*D_n)(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikx} \to \frac{f(x^+)+f(x^-)}{2}$ (if applicable).

Theorem 5.4 If f is piecewise continuous on $[-\pi,\pi]$, then $(f*\sigma_n)(x) \to f(x)$ whenever f is continuous at x, $x \in (-\pi,\pi)$. Moreover, if f is continuous and of period 2π, then $(f*\sigma_n)(x) \to f(x)$ uniformly.

- Fejér's Kernel: $\sigma_n(t) = \frac{1}{n+1}\left[\frac{\sin\frac{n+1}{2}t}{\sin\frac{t}{2}}\right]^2 \leq \frac{1}{n+1}\frac{1}{\sin^2\frac{t}{2}}$.
- Cesàro Means: $\frac{a_1+\cdots+a_n}{n}$ for a sequence $\{a_n\}$.

Theorem 5.5 If f is piecewise continuous on $[-\pi,\pi]$, then $(f*P_r)(x) \to f(x)$ as $r \to 1^-$ whenever f is continuous at x, $x \in (-\pi,\pi)$.

- Poisson's Kernel: $P_r(t) = \sum_{k\in\mathbb{Z}} r^{|k|}e^{ikt}$.
- Abel's Means: $\lim_{r\to 1}\sum_{k\in\mathbb{Z}} c_k r^{|k|}e^{ikt}$ for a series $\sum c_k e^{ikx}$.

3 Fourier Approximation

Best Approximation: Let $X = (X, \|\cdot\|)$ be a normed space and Y be a fixed subspace of X. If there exists $y_0 \in Y$ such that $\|x - y_0\| = \inf_{y\in Y}\|x-y\|$, then y_0 is called a best approximation to x out of Y.

- Existence Theorem If Y is finite dimensional, then for each $x \in X$ there exists a best approximation to x out of Y.
- Uniqueness Theorem If X is strictly convex, then for each $x \in X$ there exists at most one best approximation to x out of Y.
  - ▷ Convex: $y, z \in M \Rightarrow W = \{v = \alpha y + (1-\alpha)z | 0 \leq \alpha \leq 1\} \subseteq M$.
  - ▷ Lemma 6.2.1 The set of best approximations to x is convex.
  - ▷ Strict Convexity: $\forall x \neq y$ of norm 1 $[\|x+y\| < 2]$.
    - ∗ Hilbert space is strictly convex.
    - ∗ $C[a,b]$ is not strictly convex.
- Theorem 6.2.5 Let H be a Hilbert space and Y be any closed subspace of H, then for every $x \in H$ there is a unique best approximation to x out of Y.

Uniform Approximation: $\|x\| = \max_{t\in J}|x(t)|$, where $J = [a,b]$.

- Extremal Point: An extremal point of an $x \in C[a,b]$ is a $t_0 \in [a,b]$ such that $|x(t_0)| = \|x\|$.
- Haar Condition: A finite dimensional subspace Y of the real space $C[a,b]$ satisfies the Haar condition if every $y \in Y, y \neq 0$ has at most $n-1$ zeros in $[a,b]$, where $n = \dim Y$.
  - ▷ Lemma 6.3.3 Suppose Y satisfies the Haar condition. If for a given $x \in C[a,b]$ and a $y \in Y$ the function $x - y$ has less than $n+1$ extremal points, then y is not a best approximation to x.
- Haar Uniqueness Theorem The best approximation out of Y is unique for every $x \in C[a,b]$ iff Y satisfies the Haar condition.
  - ▷ Theorem 6.3.5 The best approximation to an $x \in C[a,b]$ out of Y_n is unique, where Y_n is the subspace containing 0 and all polynomials of degree not exceeding a fixed given n.
- Chebyshev Polynomials: The polynomial $\tilde{T}_n(t) = \frac{1}{2^{n-1}}T_n(t) = \frac{1}{2^{n-1}}\cos(n\arccos t)$ $(n \geq 1)$ is the best approximation of 0 out of all real polynomials on $[-1,1]$ of degree n and leading coefficient 1.
  - ▷ Recursive formula: $T_{n+1}(t) + T_{n-1}(t) = 2tT_n(t)$.
  - ▷ Lemma 6.4.2 Let Y be a subspace of $C[a,b]$ satisfying the Haar condition. Given $x \in C[a,b]$, let $y \in Y$ be such that $x - y$ has an alternating set of $n+1$ points, where $n = \dim Y$. Then y is the best uniform approximation to x out of Y.

Least Squares Approximation: $\|x\| = \sqrt{\langle x,x\rangle} = \sqrt{\int_a^b |x(t)|^2\,dt}$.

- $\{v_1, \cdots, v_n\}$ is a family of orthonormal vectors. For all $u \in V$,
$$\inf\{\|u - v\| : v \in \mathrm{Span}\{v_1,\cdots,v_n\} = M\} = \left\|\sum_{j=1}^n \langle u, v_j\rangle v_j - u\right\|.$$

- $\triangleright$ $\sum_{j=1}^{n} \langle u, v_j \rangle v_j = P_M u$ is the orthogonal projection of u to M.
  - $\triangleright$ $P_M u$ is independent of choice of basis.
  - $\triangleright$ $u - P_M u$ is perpendicular to M.
  - $\triangleright$ $||P_M u - P_M v|| \le ||u - v||$. This implies $P_M u$ is continuous.
- Approximation in $\mathbb{R}^n$: Given $M = \text{Span}\{a_1, \cdots, a_m\}$ which is a subspace of $\mathbb{R}^n$, let $A = [a_1, \cdots, a_m]$. The best approximation to any $b \in \mathbb{R}^n$ out of M, $A\alpha^*$, satisfies $A^\top A\alpha^* = A^\top b$.
  - $\triangleright$ Gram Determinant: The determinant of $A^\top A$ is the Gram determinant of A, denoted as $G(a_1, \cdots, a_m)$. We have
$$||b - P_M b||^2 = \frac{G(b, a_1, \cdots, a_m)}{G(a_1, \cdots, a_m)} \text{ for any vector } b \in \mathbb{R}^n.$$
- Approximation in L^2: Let $\{\varphi_k\}$ be a family of orthonormal set. If $c_k = \langle f, \varphi_k \rangle$ for all k, then for any $n \in \mathbb{N}$ and $\{\gamma_k\} \subset \mathbb{R}$, we have
$$\int_a^b \left| f(x) - \sum_{k=1}^{n} \gamma_k \varphi_k(x) \right|^2 \mathrm{d}x \ge \int_a^b \left| f(x) - \sum_{k=1}^{n} c_k \varphi_k(x) \right|^2 \mathrm{d}x.$$
- Approximation with Fourier Series: Fourier series of f is its best approximation out of the Cosine and Sine basis.

Theorem 3.14 Let $\{\varphi_k : k \in \mathbb{N}\}$ be an orthonormal basis of $L^2[a,b]$. Then for any $f \in L^2[a,b]$, $\int_a^b |f(x)|^2 \, \mathrm{d}x = \sum_{k=1}^{\infty} c_k^2$ [*Parseval's identity*], where $c_k = \int_a^b f(x)\varphi_k(x) \, \mathrm{d}x$. Note that $f = \sum_{k=1}^{\infty} c_k \varphi_k$.

- In particular, let f be a piecewise continuous function on $[0, L]$, let $a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi x}{L}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi x}{L}\right)$ be the Fourier series of f on $[0, L]$, then $\int_0^L |f(x)|^2 \, \mathrm{d}x = \frac{L}{2}\left(2a_0^2 + \sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right)\right)$.
- Let f be a piecewise continuous function on $[0, \pi]$, then $\int_0^\pi |f(x)|^2 \, \mathrm{d}x = \frac{\pi}{2} \sum_{k=1}^{\infty} b_k^2 = \frac{\pi}{2}\left(2a_0^2 + \sum_{k=1}^{\infty} a_k^2\right)$, where a_0, a_k and b_k are the Fourier cosine and sine coefficients respectively.

Theorem 3.17 Let w be a weight on a finite interval $[a,b]$, and let $f \in L_w^2[a,b]$. Then $p_n^* \in P_n$ is the least squares approximation of f out of P_n if and only if $\langle f - p_n^*, p \rangle = 0$ for all $p \in P_n$. Moreover, $p_n^*(x) = \sum_{k=0}^{n} \alpha_k^* x^k$, where
$$\begin{bmatrix} \langle 1, 1 \rangle_w & \cdots & \langle x^n, 1 \rangle_w \\ \vdots & \ddots & \vdots \\ \langle 1, x^n \rangle_w & \cdots & \langle x^n, x^n \rangle_w \end{bmatrix} \begin{bmatrix} \alpha_0^* \\ \vdots \\ \alpha_n^* \end{bmatrix} = \begin{bmatrix} \langle f, 1 \rangle_w \\ \vdots \\ \langle f, x^n \rangle_w \end{bmatrix}.$$

4 Application of Fourier Series

PDE: Separation of variables & verify.

- $u(x, 0) = 0, u(x, 1) = 2 \Rightarrow$ Let $v = u + w(y)$.
- $u(x, 0) = 1, u(x, 1) = 1 \Rightarrow$ Let $v = u - 1$.

Eigenvalue Problem: Consider the ODE $L(y) = f(x)$. Find y_k such that $L(y_k) = \lambda y_k$ has non-trivial solutions for some $\lambda \in \mathbb{R}$.

- $y(0) = 0, y(\pi) = 1 \Rightarrow$ Let $v = y - \frac{x}{\pi}$.

Sturm-Liouville Problem: Consider the self-adjoint DE $(p(x)y')' - q(x)y + \lambda r(x)y = 0$ on $[a,b]$ with boundary conditions $a_0 y(a) + a_1 y'(a) = 0$ and $b_0 y(b) + b_1 y'(b) = 0$. Find λ and corresponding non-trivial ϕ_λ.

- Regular: $p, r > 0$ on $[a,b]$ & p, p', q, r are continuous.
- Spectrum: Set of all eigenvalues of a regular SL problem.
- Theorems
  - If ϕ_1 and ϕ_2 are eigenfunctions corresponding to the same eigenvalue, then $\phi_1 = k\phi_2$ for some k.
  - If $\lambda_1 \ne \lambda_2$, then ϕ_{λ_1} and ϕ_{λ_2} are linearly independent. Also,
$$\int_0^\pi \phi_{\lambda_1}(x)\phi_{\lambda_2}(x) \, \mathrm{d}x = 0 \text{ or } \int_0^\pi \phi_{\lambda_1}(x)\phi_{\lambda_2}(x) r(x) \, \mathrm{d}x = 0.$$
  - All eigenvalues are real.
  - Infinite eigenvalues $\lambda_1 < \cdots < \lambda_n < \cdots$ where $\lim_{n\to\infty} \lambda_n \to \infty$.

Fourier-Legendre Series: $\{P_k(x) : k \in \mathbb{N}\}$ is a family of orthogonal functions on $[-1, 1]$. The Fourier-Legendre series of a function f is
$$f(x) = \sum_{k=0}^{\infty} c_k P_k(x), \text{where } c_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) P_k(x) \, \mathrm{d}x.$$
Here $P_k(x) = \frac{1}{2^k k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^2 - 1)^k \in \left\{1, x, \frac{1}{2}\left(3x^2 - 1\right), \cdots\right\}$.

Green Function: Consider the second-order self-adjoint DE $L[y] = f$. A solution to this function would be $2\pi(G * f)$, where the Green function G is the solution to $L[G] = \delta_0$.

- $y = \sum_{k\in\mathbb{Z}} \frac{e^{ikx}}{2-k^2}$ is a solution to $y'' + 2y = \delta_0$.

Trigometric Identity

$\cos x \cos y = \frac{\cos(x-y)+\cos(x+y)}{2}$ $\quad$ $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$

$\sin x \sin y = \frac{\cos(x+y)-\cos(x-y)}{2}$ $\quad$ $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$

$\sin x \cos y = \frac{\sin(x+y)+\sin(x-y)}{2}$

$\cos^2 x = \frac{1+\cos 2x}{2}$ $\quad$ $\sin^2 x = \frac{1-\cos 2x}{2}$

Inner Product Space

Inner product: $\langle \cdot, \cdot \rangle$ is said to be an inner product on a real vector space V if for all $f, g, h \in V$, we have (1) $\langle f, g \rangle = \langle g, f \rangle$ (symmetric); (2) $\langle f, g + ch \rangle = \langle f, g \rangle + c\langle f, h \rangle$ and $\langle f + ch, g \rangle = \langle f, g \rangle + c\langle h, g \rangle$ (bilinear); (3) $\langle f, f \rangle \ge 0$ with equality only when $f = 0$ (positive definite).

- Orthogonal: A set $\mathcal{F}$ in a vector space with inner product $\langle \cdot, \cdot \rangle$ is said to be orthogonal if $\langle f, g \rangle = 0$ for all $f, g \in \mathcal{F}, f \ne g$.
  - $\triangleright$ An orthogonal family of vectors is linearly independent.
- Orthonormal: Orthogonal & $\langle f, f \rangle = 1$ for all $f \in \mathcal{F}$.
- Orthonormal Basis: An orthonormal set $\{e_1, e_2, \cdots\}$ in a vector space V is an orthonormal basis of V if for any $f \in V$, there exists unique $\{c_k\} \subset \mathbb{R}$ such that $f = \sum_{k=1}^{\infty} c_k e_k$. It is complete.

Norm: $||x|| = \sqrt{\langle x, x \rangle}$.

- Cauchy-Schwartz Inequality $|\langle x, y \rangle| \le ||x|| \, ||y||$.
- Parallelogram Rule $||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$.
- Pythagorean Law $\left|\left|\sum_{j=1}^{k} v_j\right|\right|^2 = \sum_{j=1}^{k} ||v_j||^2$ if $\langle v_j, v_l \rangle = 0$ for all $j \ne l$.

Hilbert space: Complete inner product space.

- Every Hilbert space has a maximal (complete) orthonormal set.

Bessel's Inequality Let $\mathcal{F}$ be a family of orthonormal vectors in V,
$$\sum_{v_\alpha \in \mathcal{F}} |\langle v, v_\alpha \rangle|^2 \le ||v||^2 \text{ for all } v \in V.$$

Parseval's Identity If $\mathcal{F}$ is complete and hence an orthonormal basis,
$$\sum_{v_\alpha \in \mathcal{F}} |\langle v, v_\alpha \rangle|^2 = ||v||^2 \text{ for all } v \in V.$$

Functional Analysis

Bessel's Inequality If $\{\psi_k\}$ is orthonormal on $[a,b]$, then
$$\langle f, f \rangle \ge \sum_{k=1}^{n} |(f, \psi_k)|^2 \text{ for all } n.$$

- Let f be a piecewise continuous function on $[0, T]$, then
$$\lim_{n\to\infty} \int_0^T f(x) \sin\frac{(2n+1)\pi x}{2T} \, \mathrm{d}x = 0.$$

Theorem 4.10 (1) If f_n is continuous on an interval I for each $n \in \mathbb{N}$ and $\sum f_n(x)$ converges uniformly to f on I, then f is continuous.
(2) Let f_n be differentiable functions on an interval J for each $n \in \mathbb{N}$ such that $\sum f_n'(x)$ converges uniformly on all bounded subintervals of J. If $\exists x_0 \in J$ such that $\sum f_n(x_0)$ converges, then the series $\sum f_n(x)$ converges uniformly to a differentiable function f on any bounded subintervals of J and $f'(x) = \sum f_n'(x)$ on J.

Cauchy Criterion A series $\sum f_n(x)$ converges uniformly on I iff $\forall \varepsilon > 0 \, \exists K(\varepsilon)$ s.t. $|f_m(x) + f_{m-1}(x) + \cdots + f_{n+1}(x) < \epsilon|$ for all $x \in I$ and $m > n \ge K$.

Weierstrass M-test Let $|f_n(x)| \le M_n$ for all $x \in I, M_n \in \mathbb{R}$ for each $n \in \mathbb{N}$ and $\sum M_n < \infty$. The series $\sum f_n(x)$ converges uniformly on I.

Minkowski Inequality
$$\left(\sum_{k=1}^{\infty} |a_k + b_k|^2\right)^{\frac{1}{2}} \le \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} |b_k|^2\right)^{\frac{1}{2}}$$
$$\left(\int_a^b |f + g|^2\right)^{\frac{1}{2}} \le \left(\int_a^b |f|^2\right)^{\frac{1}{2}} + \left(\int_a^b |g|^2\right)^{\frac{1}{2}}$$

Abel's Lemma Let (a_n) and (b_n) be sequences and let $S_n = \sum_{k=1}^{\infty} b_k$ be the sequence of partial sums with $S_0 = 0$. Then for $m > n \in \mathbb{N}$,
$$\sum_{k=n+1}^{m} a_k b_k = a_m S_m - a_{n+1} S_n + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) S_k.$$

Dirichlet's Test Let (a_n) be a decreasing sequence of real numbers that converge to 0 and $\exists M > 0$ such that $\left|\sum_{k=1}^{n} b_k\right| \le M$ for all $n \in \mathbb{N}$. Then the series $\sum a_k b_k$ converges.

Abel's Test Let (a_n) be a convergent monotone sequence and let $\sum b_k$ converge. Then the series $\sum a_k b_k$ converges.