MA4229 Fourier Analysis and Approximation

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1 Fourier Series

Piecewise continuity: A function f is said to be piecewise continuous on $[a,b]$ if it has a portition $a=x_0<x_1<\cdots<x_n=b$ such that f is uniformly continuous on each interval (x_{i-1}, x_i) for $i=1,\cdots,n$.
- $f(a^+)$, $f(b^-)$, $f(x_i^+)$, $f(x_i^-)$ exist for all $i=1,\cdots,n-1$

Piecewise smooth: A function f is said to be piecewise smooth on $[a,b]$ if both f' and f are piecewise continuous on $[a,b]$.

Fourier Cosine series: If f is piecewise continuous on $[0,\pi]$, then

$$f(x)=a_0+\sum_{k=1}^{\infty} a_k\cos(kx) \begin{cases} a_0=\frac{1}{\pi}\int_0^{\pi} f(x)\,dx \\ a_k=\frac{2}{\pi}\int_0^{\pi} f(x)\cos(kx)\,dx \end{cases}$$

Fourier Sine series: If f is piecewise continuous on $[0,\pi]$, then

$$f(x)=\sum_{k=1}^{\infty} b_k\sin(kx), \quad b_k=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,dx$$

Fourier series: If f is piecewise continuous on $[-\pi,\pi]$, then

$$f(x)=a_0+\sum_{k=1}^{\infty} a_k\cos(kx)+\sum_{k=1}^{\infty} b_k\sin(kx) \begin{cases} a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \\ b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \end{cases}$$

If instead f is piecewise continuous on $[-L,L]$,

$$f(x)=a_0+\sum_{k=1}^{\infty} a_k\cos\left(\frac{2k\pi x}{L}\right)+\sum_{k=1}^{\infty} b_k\sin\left(\frac{2k\pi x}{L}\right) \begin{cases} a_0=\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\,dx \\ a_k=\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\left(\frac{2k\pi x}{L}\right)dx \\ b_k=\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\left(\frac{2k\pi x}{L}\right)dx \end{cases}$$

Integration by parts:

① When f is a piecewise continuous fn. on $[a,b]$, then $\int_a^b f(x)\,dx=\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} f(x)\,dx=\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} g_j(x)\,dx$.

② When P is a polynomial of degree less than m and f is continuous, then

$$\int Pf\,dx = P\underbrace{F_1}_{\text{antiderivative of } f}-P'F_2+P''F_3-\cdots+\cdots+(-1)^m P^{(m)}F_{m+1}+C$$

③ When P is a polynomial or other nice function and f is <u>piecewise</u> continuous, then we will do ① before ②.

2 Inner Products & Best Approximation

Inner product: $\langle\cdot,\cdot\rangle$ is said to be an inner product on a real vector space V if $\forall f,g,h\in V$, we have (1) $\langle f,g\rangle=\langle g,f\rangle$; (2) $\langle f,g+ch\rangle=\langle f,g\rangle+c\langle f,h\rangle$ and $\langle f+ch,g\rangle=\langle f,g\rangle+c\langle h,g\rangle$ (3) $\langle f,f\rangle\geq 0$ with equality only when $f=0$.

Orthogonal: A set F in a vector space with inner product $\langle\cdot,\cdot\rangle$ is said to be orthogonal if $\langle f,g\rangle=0$ for all $f,g\in F$, $f\neq g$.

Orthonormal: orthogonal & $\langle f,f\rangle=1$ for all $f\in F$.

Norm $\|x\|=\sqrt{\langle x,x\rangle}$.

Cauchy-Swartz Inequality: $|\langle x,y\rangle|\leq\|x\|\,\|y\|$.

Parallelogram Rule: $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$.

Pythagorean Law: $\|\sum_{j=1}^{k} v_j\|^2=\sum_{j=1}^{k}\|v_j\|^2$ if $\langle v_j,v_i\rangle=0$ for all $j\neq i$.

Hilbert space: A complete inner product space is called a Hilbert space.

Proposition 3.6: (1) An orthogonal family of vectors is linearly independent.
(2) Every Hilbert space has a maximal orthonormal set.

Orthonormal basis: An orthonormal set $\{e_1,e_2,\cdots\}$ in a vector space V is said to be an orthonormal basis of V if for any $f\in V$, there exists unique $\{c_k\}\subset\mathbb{R}$ such that $f=\sum_{k=1}^{\infty} c_k e_k$. It is a complete orthonormal set.

Bessel's Inequality: Let F be a family of orthonormal vectors in V, then $\sum_{v_\alpha\in F}|\langle v,v_\alpha\rangle|^2\leq\|v\|^2$ for all $v\in V$.

Parseval's Inequality: If F is also complete and hence an orthonormal basis, then $\sum_{v_\alpha\in F}|\langle v,v_\alpha\rangle|^2=\|v\|^2$ for all $v\in V$.

Best Approximation by family of orthonormal vectors:

Let $\{v_1,\cdots,v_n\}$ be a family of orthonormal vectors, then for all $u\in V$,

$$\inf\{\|u-v\|: v\in\operatorname{Span}\{v_1,\cdots,v_n\}=M\}=\Big\|\underbrace{\sum_{j=1}^{n}\langle u,v_j\rangle v_j}_{=P_M u\text{ : 'orthonormal projection'}}-u\Big\|$$

- P_M is independent of choice of basis
- $u-P_M u$ is perpendicular to M
- $\|P_M u-P_M v\|\leq\|u-v\|$. This implies P_M continuous

Least Squares Approximation in $\mathbb{R}^n$:

Given $M=\operatorname{span}\{a_1,\cdots,a_m\}$ which is a subspace of $\mathbb{R}^n$, we want to find for any $b\in\mathbb{R}^n$ a corresponding $v\in M$ s.t. $\|v-b\|=\inf_{\tilde v\in M}\{\tilde v-b\}$.

Let $A=[a_1\cdots a_m]$, then we can write the problem as $\|A\alpha^*-b\|=\min_{\alpha\in\mathbb{R}^m}\{A\alpha-b\}$

We look for the best approximation $P_M b=A\alpha^*=\sum_{i=1}^{m}\alpha_i^* a_i$. Since $b-P_M b$ is orthogonal to M, we have $\forall x\in\mathbb{R}^m$ $\langle b-A\alpha^*,Ax\rangle=0$. Hence $\langle A^T b,x\rangle=\langle b,Ax\rangle=\langle A\alpha^*,Ax\rangle=\langle A^T A\alpha^*,x\rangle$. Hence $A^TA\alpha^*=A^Tb$.

Gram determinant: Let $A=[a_1\cdots a_m]$. The determinant of A^TA is the Gram determinant of A, denoted as $G(a_1,\cdots,a_m)$. For any vector $b\in\mathbb{R}^n$, we have $\|b-P_M b\|^2=\frac{G(b,a_1,\cdots,a_m)}{G(a_1,\cdots,a_m)}$.

Best approximation in L^2:

Let $\{\varphi_k\}$ be a family of orthonormal set. If $c_k=\langle f,\varphi_k\rangle$ for all k, then for any $n\in\mathbb{N}$ and $\{\gamma_k\}\in\mathbb{R}$, we have

$$\int_a^b\Big|f(x)-\sum_{k=1}^{n}\gamma_k\varphi_k(x)\Big|^2dx\geq\int_a^b\Big|f(x)-\sum_{k=1}^{n}c_k\varphi_k(x)\Big|^2dx.$$

Best approximation for Fourier series: The Fourier series coefficients a_0, $\{a_k\}$, $\{b_k\}$ can minimize $\int_0^L\Big|a_0'+\sum_{k=1}^{\infty}a_k'\cos\left(\frac{2k\pi x}{L}\right)+\sum_{k=1}^{\infty}b_k'\cos\left(\frac{2k\pi x}{L}\right)-f(x)\Big|dx$

Theorem 3.14. Let $\{\varphi_k: k\in\mathbb{N}\}$ be an orthonormal basis of $L^2[a,b]$. Then for any $f\in L^2[a,b]$, $\int_a^b|f(x)^2|\,dx=\sum_{k=1}^{\infty}c_k^2$ (Parseval's equation/identity) where $c_k=\int_a^b f(x)\varphi_k(x)\,dx$. Note that $f=\sum_{k=1}^{\infty}c_k\varphi_k$.

In particular, let f be a piecewise continuous function on $[0,L]$, let $a_0+\sum_{k=1}^{\infty}a_k\cos\left(\frac{2k\pi x}{L}\right)+\sum_{k=1}^{\infty}b_k\sin\left(\frac{2k\pi x}{L}\right)$ be the Fourier series of f on $[0,L]$, then $\int_0^L|f(x)|^2dx=\frac{L}{2}\left(2a_0^2+\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right)$.

Similarly, if f is piecewise continuous on $[0,\pi]$ $\begin{cases}\int_0^{\pi}|f(x)|^2dx=\frac{\pi}{2}\sum_{k=1}^{\infty}b_k^2. \\ \int_0^{\pi}|f(x)^2|\,dx=\frac{\pi}{2}\left(2a_0^2+\sum_{k=1}^{\infty}a_k^2\right)\end{cases}$

Theorem 3.17: Let w be a weight on a finite interval $[a,b]$, and let $f\in L_w^2[a,b]$. Then $p_n^*\in P_n$ is the least squares approximation of f out of P_n if and only if $\langle f-p_n^*,p\rangle_w=0$ $\forall p\in P_n$.

Moreover, $p_n^*(x)=\sum_{k=0}^{n}\alpha_k^* x^k$, where

$$\begin{pmatrix}\langle 1,1\rangle_w & \cdots & \langle x^n,1\rangle_w \\ \vdots & & \\ \langle 1,x^n\rangle_w & \cdots & \langle x^n,x^n\rangle_w\end{pmatrix}\begin{pmatrix}\alpha_0^* \\ \vdots \\ \alpha_n^*\end{pmatrix}=\begin{pmatrix}\langle f,1\rangle_w \\ \vdots \\ \langle f,x^n\rangle_w\end{pmatrix}.$$

3 Convergence of Fourier Series

Theorem 4.1. Let f be a piecewise smooth function on $[0,L]$. Then the Fourier series $a_0+\sum_{k=1}^{\infty}a_k\cos\left(\frac{2k\pi x}{L}\right)+\sum_{k=1}^{\infty}b_k\sin\left(\frac{2k\pi x}{L}\right)$ of f converges to $\frac{f(x^+)+f(x^-)}{2}$ for all $x\in(0,L)$. Moreover, if $x=0$ or L, then the series converges to $\frac{f(0^+)+f(L^-)}{2}$.
- If f is continuous at $x_0\in(0,L)$, the Fourier series converge to $f(x_0)$.

Corollary 4.4. Assume f is a piecewise continuous function on $[0,L]$, and $a_0+\sum a_k\cos+\sum b_k\sin$ is the Fourier series of f, then $\lim_{k\to\infty}a_k=0=\lim_{k\to\infty}b_k$.
- It is enough to assume only $\int_0^L|f(x)|\,dx<\infty$.

Fact 1 (Bessel's Inequality): $\{\psi_k\}$ is orthonormal on $[a,b]$, then

$$\langle f,f\rangle\geq\sum_{k=1}^{n}|(f,\psi_k)|^2 \text{ for all } n.$$

Fact 2. Let f be a piecewise continuous function on $[0,T]$, then

$$\lim_{n\to\infty}\int_0^T f(x)\sin\frac{(2n+1)\pi x}{2T}\,dx=0$$

Dirichlet Kernel: $D_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos kx=\frac{\sin\left(\frac{2n+1}{2}x\right)}{2\sin\left(\frac{x}{2}\right)}$ when $x\neq 2k\pi$, $k\in\mathbb{Z}$.
Note that $\int_0^{\pi}D_n(x)\,dx=\frac{\pi}{2}$.

Theorem. Let f be a piecewise smooth function on $[0,L]$. Then its Fourier sine and cosine series converge to $\frac{f(x^+)+f(x^-)}{2}$ if $x\in(0,L)$.

Definition 4.9. A (infinite) series of functions $\sum f_n(x)$ is said to converge to f on a set A_0 if its sequence of initial sums $\left(\sum_{k=1}^{n}f_k(x)\right)_{n=1}^{\infty}$ converges to $f(x)$ for every $x\in A_0$. If $\left(\sum_{k=1}^{n}f_k(x)\right)_{n=1}^{\infty}$ converges uniformly to $f(x)$ on A_0, then we say $\sum_{n=1}^{\infty}f_n(x)$ converges uniformly to f on A_0. If $\left(\sum_{k=1}^{n}|f_k(x)|\right)_{n=1}^{\infty}$ converges for all $x\in A_0$, we say the series $\sum_{n=1}^{\infty}f_n(x)$ converges absolutely on A_0.

Theorem 4.1a ① If f_n is continuous on an interval I for each $n \in \mathbb{N}$ and $\sum f_n(x)$ converges uniformly to f on I, then f is also continuous.

② Let f_n be differentiable functions on an interval J for each $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on all bounded subintervals of J. If $\exists x_0 \in J$ such that $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then the series $\sum f_n(x)$ converges uniformly to a differentiable function f on any bounded subintervals of J and $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$ on J.

Cauchy criterion: A series $\sum f_n(x)$ converges uniformly on I if and only if given any $\varepsilon > 0$, $\exists K = K(\varepsilon)$ s.t. $|f_n(x) + f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon$ for all $x \in I$, $m > n \geq K$.

Weierstrass M-test: Let $|f_n(x)| \leq M_n$ for all $x \in I$, $M_n \in \mathbb{R}$ for each $n \in \mathbb{N}$ and $\sum M_n < \infty$. Then the series $\sum f_n(x)$ converges uniformly on I.

Minkowski's inequality
$$\begin{cases} \left(\sum_{k=1}^{\infty} |a_k + b_k|^2\right)^{\frac{1}{2}} \leq \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} |b_k|^2\right)^{\frac{1}{2}} \\ \left(\int_a^b |f+g|^2\right)^{\frac{1}{2}} \leq \left(\int_a^b |f|^2\right)^{\frac{1}{2}} + \left(\int_a^b |g|^2\right)^{\frac{1}{2}} \end{cases}$$

Theorem 4.11. Let f be a continuous function of period 2π such that its derivative f' is piecewise continuous on $[-\pi, \pi]$. Then the Fourier series of f, $a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$ is differentiable at each point $x_0 \in (-\pi, \pi)$ at which the second derivative f'' exists: $f'(x_0) = \sum_{k=1}^{\infty} k(-a_k \sin kx_0 + b_k \cos kx_0)$.

Abel's Lemma: Let (a_n) and (b_n) be sequences and let $S_n = \sum_{k=1}^{n} b_k$ be the sequence of partial sums with $S_0 = 0$. Then
$$\sum_{k=n+1}^{m} a_k b_k = a_m S_m - a_{n+1} S_n + \sum_{k=n+1}^{m-1} (a_k - a_{k+1}) S_k$$
for $m > n$, $m, n \in \mathbb{N}$.

Dirichlet's Test: Let (a_n) be a decreasing sequence of real numbers that converge to 0 and $\exists M > 0$ s.t. $\left|\sum_{k=1}^{N} b_k\right| \leq M$ for $N \in \mathbb{N}$. Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

Abel's Test: Let (a_n) be a convergent monotone sequence and let $\sum_{k=1}^{\infty} b_k$ converges. Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proposition 4.12. If $a_k \searrow 0$, the Fourier sine series $\sum_{k=1}^{\infty} a_k \sin kx$ converges uniformly on $[\delta, \pi - \delta]$ for all $\delta > 0$ ($\delta < \frac{\pi}{2}$).

Theorem 4.13. If f is a continuous function of period 2π such that f' is piecewise continuous on $[-\pi, \pi]$, then $k a_k$, $k b_k \to 0$ as $k \to \infty$.

Fourier series of complex-valued functions:
$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx} \quad \text{with } c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx.$$

- The family $\langle e^{ikx} : k \in \mathbb{Z} \rangle$ is orthogonal if w.r.t. $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)}\, dx$
- If f is piecewise continuous on $[0, 2\pi]$, then $\sum_{k=0}^{\infty} \hat{f}(k) z^k$ converges on the open unit disk $\{|z| < 1\}$ and hence analytic on this open disk.
- Define $\tilde{f}(z) = \sum_{k \in \mathbb{Z}} \hat{f}(k) z^k$, then $\tilde{f}(e^{ix}) = f(x)$ if f is piecewise smooth, continuous and of period 2π.

Definition 5.2. Let f and g be both periodic (of period 2π) piecewise continuous functions on $[-\pi, \pi]$. Then we define its convolutions as
$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-y) g(y)\, dy.$$
Then $\widehat{f * g}(n) = \hat{f}(n) \hat{g}(n)$.

Proposition 5.3. (1) $f * g = g * f$ (2) $(f * g) * h = f * (g * h)$.
(3) $(\alpha f_1 + f_2) * g = \alpha(f_1 * g) + f_2 * g$. (4) $f * g$ is continuous.

Lemma: If f is a periodic function of period L and f is piecewise continuous on $[0, L]$, then $\int_0^L f(x-y)\, dy = \int_0^L f(y)\, dy$ for any $x \in \mathbb{R}$.

Note that it is clear that $\int_0^L f(x+y)\, dy = \int_0^L f(y)\, dy = \int_{-\frac{L}{2}}^{\frac{L}{2}} f(y)\, dy$ for any $x \in \mathbb{R}$

Fubini's theorem: For a function F on $[a,b] \times [c,d]$,
$$\int_a^b \int_c^d F(x,y)\, dy\, dx = \int_c^d \int_a^b F(x,y)\, dx\, dy$$

Cesàro means: For $\{a_n\}$, find the limit of $\frac{a_1 + \cdots + a_n}{n}$.

Fejér's kernel:
$$\sigma_n(t) = \frac{1}{n+1} \left[\frac{\sin \frac{n+1}{2} t}{\sin \frac{t}{2}} \right]^2 \leq \frac{1}{n+1} \frac{1}{\sin^2 \frac{\delta}{2}}$$

Theorem 5.4. If f is piecewise continuous on $[-\pi, \pi]$, then $f * \sigma_n(x) \to f(x)$ whenever f is continuous at $x \in (-\pi, \pi)$. Moreover, if f is continuous and periodic (2π), then uniformly.

Theorem 5.5. (Poisson kernel) Let $P_r(t) = \sum_{k \in \mathbb{Z}} r^{|k|} e^{ikt}$ for $|r| < 1$ and let f be a piecewise continuous function on $[-\pi, \pi]$. Then $f * P_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) \left[\sum_{k \in \mathbb{Z}} r^{|k|} e^{ikt} \right] dt \to f(x)$ as $r \to 1^-$, whenever f is continuous at x, $x \in (-\pi, \pi)$.

Existence Theorem: If Y is a finite dimensional subspace of a normed space $X = (X, \|\cdot\|)$, then for each $x \in X$ there exists a best approximation to x out of Y.

Lemma (Convexity): In a normed space $(X, \|\cdot\|)$, the set M of best approximations to a given point x out of a subspace Y of X is convex.

Strictly convex norm: A norm such that $\forall x, y$ of norm 1, $\|x + y\| < 2$.

Uniqueness Theorem: In a strictly convex normed space X there is at most one best approximation to an $x \in X$ out of a given subspace Y.

Lemma. (a) Hilbert space is strictly convex.
(b) $C[a,b]$ is not strictly convex.

Theorem: For every given x in a Hilbert space H and every given closed subspace Y of H there is a unique best approximation to x out of Y.

Extremal point: An extremal point of an x in $C[a,b]$ is a $t_0 \in [a,b]$ such that $|x(t_0)| = \|x\|$.

Haar condition: A finite dimensional subspace Y of the real space $C[a,b]$ satisfies Haar condition if every $y \in Y$, $y \neq 0$ has at most $n-1$ zeros in $[a,b]$, where $n = \dim(Y)$.

$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos kx, \frac{1}{\sqrt{\pi}} \sin kx : k \in \mathbb{N} \right\}$ orthonormal basis of $L^2(-\pi, \pi)$

$\left\{ \frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}} \cos kx : k \in \mathbb{N} \right\} \left\{ \sqrt{\frac{2}{\pi}} \sin kx : k \in \mathbb{N} \right\}$ $L^2(0, \pi)$

Differential Equation: $y' = ay \Rightarrow y = Ce^{at}$
$$ay'' + by' + c = 0 \Rightarrow \begin{cases} r_1 \neq r_2 \Rightarrow y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ r_1 = r_2 \Rightarrow y = (c_1 + c_2 t) e^{rt} \\ \lambda \pm \mu i \Rightarrow y = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t)) \end{cases}$$

Trigonometric Identity:

$\cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2}$ $\quad \cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$

$\sin x \sin y = \frac{\cos(x+y) - \cos(x-y)}{2}$ $\quad \sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$

$\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}$

$\cos^2 x = \frac{1 + \cos 2x}{2}$ $\quad \sin^2 x = \frac{\cos 2x - 1}{2}$

Useful facts:
$$\int_0^{2\pi} \sin nx \sin mx\, dx = \pi \delta(m-n) \begin{cases} = 0, \text{ if } m \neq n \\ = 1, \text{ if } m = n \end{cases}$$

If a series $\sum_{n=1}^{\infty} f(n)$ converges, then $\lim_{n \to \infty} f(n) \to 0$.

Let $\{\phi_k : k \in \mathbb{N}\}$ be an orthonormal basis. Then if $f = \sum_{k=1}^{\infty} c_k \phi_k$, then $c_k = \langle f, \phi_k \rangle$.

Good luck!