**MA4254 Discrete Optimisation**

**Midterm Examination Helpsheet**

AY2024/25 Semester 1 · Prepared by Tian Xiao *@snoidetx*

## 1 Introduction

**Mixed Integer Linear Programmes:** (standard form)

$$\begin{aligned} \min_{\mathbf{x},\mathbf{y}} \quad & \mathbf{c}^\top\mathbf{x} + \mathbf{d}^\top\mathbf{y} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{b} \\ & \mathbf{x},\mathbf{y} \geq \mathbf{0}, \mathbf{x} \in \mathbb{Z}^n, \mathbf{y} \in \mathbb{R}^k. \end{aligned}$$

**Formulation and Examples:**

item $1, 2, \cdots, n$
weight $w_1, w_2, \cdots, w_n$
cost $c_1, c_2, \cdots c_n$

$$\max_x \sum_{i=1}^n c_i x_i$$
$$\text{s.t.} \sum_{i=1}^n w_i x_i \leq B, \quad \text{total capacity}$$
whether we pick $i$ ← $x_i \in \{0, 1\}$.

Knapsack

item $1, 2, \cdots, n$
size $a_1, a_2, \cdots, a_n$ $(a_i)$
container $1, 2, \cdots, m$
size $b_1, b_2, \cdots, b_m$ $(b_j)$
binary feasibility problem: whether a feasible point exists.
$\sum_{j=1}^m x_{ij} = 1 \ \forall i$ $\quad \sum_{i=1}^n a_i x_{ij} \leq b_j \ \forall j$
$x_{ij} \leq y_j \ \forall i, j$ $\quad \sum_{i=1}^m b_i y_i \leq Q$ (ship capacity) $\quad x_{i,j} \in \{0,1\}$, $y_i \in \{0,1\}$

Shipping containers

Service locations $1, 2, \cdots, m$
customers $1, 2, \cdots, m$
$c_j$: cost for location $j$ to be switched on
$d_{ij}$: cost for customer $i$ to be serviced by facility $j$.

$$\min_{x,y} \sum_{j=1}^n c_j y_j + \sum_{j=1}^n \sum_{i=1}^m d_{ij} x_{ij}$$
$$\text{s.t.} \sum_{j=1}^n x_{ij} = 1 \ \forall i, \ x_{ij} \leq y_j \ \forall i, j$$
$$x_{ij} \in \{0,1\}, \ y_j \in \{0,1\}.$$

Facility location (relax)

item $1, 2, \cdots, n$
subset $F_1, F_2, \cdots, F_n$ (disjoint)
Find collection of subsets with highest value.
Incidence matrix $A_{ij} = \begin{cases} 1, & j \in F_i \\ 0, & \text{otherwise} \end{cases}$

$x_j$: whether $F_j$ is chosen →
$$\max_x c^T x$$
$$\text{s.t.} \ A^T x \leq 1, \ x \in \{0,1\}^m.$$

Packing

item $1, 2, \cdots, n$
subset $F_1, F_2, \cdots, F_n$
Optimise objective when all items are covered
$A_{ij} = \begin{cases} 1, & j \in F_i \\ 0, & \text{otherwise} \end{cases}$

$x_j$: whether $F_j$ is chosen ←
$$\min_x c^T x$$
$$\text{s.t.} \ A^T x \geq 1, \ x \in \{0,1\}^m$$

Covering

item $1, 2, \cdots, n$
subset $F_1, F_2, \cdots, F_n$
Find partition that maximize objective
$A_{ij} = \begin{cases} 1, & j \in F_i \\ 0, & \text{otherwise} \end{cases}$

$x_j$: whether $F_j$ is chosen.
$$\max c^T x$$
$$\text{s.t.} \ A^T x = 1, \ x \in \{0,1\}.$$

Partitioning

One of $a^T x \geq b$ and $(a')^T x \geq b'$ needs to be satisfied.
$$\max_x c^T x$$
$$\text{s.t.} \ a^T x \geq yb$$
$$(a')^T x \geq (1-y) b$$
$$y \in (0,1), \ x \geq 0.$$

Disjunction

At least $k$ inequalities from $\{x : a_i^T x \geq b_i\}$ need to be satisfied:
$$\max_x c^T x$$
$$\text{s.t.} \ a_i^T x \geq b_i y_i,$$
$$\sum_i y_i \geq k,$$
$$y_i \in \{0,1\}, \ x \geq 0$$

More disjunction

Assume $a_i^T x \geq r \ \forall x \geq 0, x \in C$,
we want $x \geq 0$, $x \in C$
at least $k$ from $\{x : a_i^T x \geq b_i\}$.
let $y_i := \begin{cases} 1, & \text{if constraint } i \text{ is satisfied} \\ 0, & \text{otherwise} \end{cases}$
Then the feasible region is
$$a_i^T x \geq y_i (b_i - r) + r$$
$$\sum_i y_i \geq k$$
$$y_i \in \{0,1\} \ x \geq 0, \ x \in C.$$

Even more disjunction

$nnz(x)$: no. of non-zero entries in $x$.
$$\min_{x \in [-M, M]^n} nnz(x) \ \text{s.t.} \ Ax \leq b$$
⇓ MILP
$$\min_{x,z} \sum_{i=1}^n z_i \ \text{s.t.} \ Ax \leq b$$
$$\forall i, \ -M z_i \leq x_i \leq M z_i$$
$$x \in \mathbb{R}^n, \ z \in \{0,1\}^n.$$

Non-linear to linear

People $1, 2, \cdots, n$
Job $1, 2, \cdots, n$
cost of person $j$ doing job $i$ : $C_{ij}$
$$\min \sum_{j=1}^n \sum_{i=1}^n C_{ij} x_{ij} \Rightarrow \text{whether person } j \text{ does job } i$$
$$\text{s.t.} \sum_{j=1}^n x_{ij} = 1 \ \forall i$$
$$\sum_{i=1}^n x_{ij} = 1 \ \forall j$$
$$x_{ij} \in \{0,1\} \ \forall i, j \quad i, j \in \{1, \cdots, n\}$$

Job scheduling

$G = (V, E)$
k-coloring: Each vertex has a color s.t. adjacent nodes have different color.
$$\min_{x,y} \sum_j y_j \longrightarrow \text{if color } j \text{ required}$$
$$\text{s.t.} \sum_j x_{ij} = 1 \longrightarrow \text{if node } i \text{ to color } j$$
$$x_{uj} + x_{vj} \leq y_j \quad \forall (u,v) \in E, \ \forall j$$
$$x_{ij}, y_j \in \{0,1\}$$

$k$-coloring

Optimal route for driver to traverse $n$ cities and return.
Graph $G = \{V, E\}$ $S \subseteq \{1, 2, \cdots, n\}$
$\delta(S)$ = subset of edges from $S$ to $S'$.
$$\min_{\{x_e\}} \sum_{e \in E} c_e x_e$$
each city once ⇐ $\sum_{e \in \delta(i)} x_e = 2 \quad \forall i \in V$
no △△ sub-tour ⇐ $\sum_{e \in \delta(S)} x_e \geq 2 \quad \forall \emptyset \subset S \subset V$
$$x_e \in \{0,1\}.$$

TSP

Optimal route for driver to traverse $n$ cities and return, but $c_{ij} \neq c_{ji}$.
Graph $G = (V, E)$. $S \subseteq \{1, 2, \cdots, n\}$
$\delta^+(S)$ : $S$ to $S'$ $\quad \delta^-(S)$ : $S'$ to $S$.
$$\min \sum_{a \in A} c_a x_a$$
$$\text{s.t.} \sum_{a \in \delta^+(i)} x_a = 1 \ \forall i \in V$$
$$\sum_{a \in \delta^-(i)} x_a = 1 \ \forall i \in V \quad x_a \in \{0,1\}$$
$$\sum_{a \in \delta^-(S)} x_a \geq 1 \ \forall S, 2 \leq |S| \leq |V| - 1$$

Asymmetric TSP

**Thm. 3.1** Suppose $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \geq \mathbf{0}\}$ is a non-empty polyhedron, and let $\mathbf{x} \in P$. The following are equivalent:

- $\mathbf{x}$ is a vertex;
- $\mathbf{x}$ is an extreme point;
- $\mathbf{x}$ is a basic feasible solution with non-negative entries.

## 2 Geometry of Linear Programming

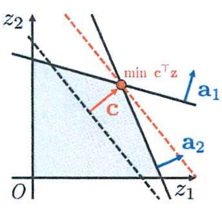
**Standard Form:** Every LP can be converted to the following *standard form* (e.g., by adding slack variables):

$$\begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{c}^\top\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

- Maximisation with $\mathbf{c}$ is minimisation with $-\mathbf{c}$.
- $a_i x_i \leq b \Leftrightarrow a_i x_i + s_i = b; s_i \geq 0$;
- $x$ uncontrained $\Leftrightarrow x^+ - x^-; x^+, x^- \geq 0$;

**Polyhedron:** $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \geq \mathbf{0}\}$.

- Convex sets defined by linear inequalities (equalities).
- **Polytype:** Bounded polyhedron.
- **Extreme point:** Let $S \subseteq \mathbb{R}^n$ be a set (polyhedron or otherwise). We say that $\mathbf{y} \in S$ is an *extreme point* of $S$ if, whenever $\mathbf{y} = \theta\mathbf{z}_1 + (1-\theta)\mathbf{z}_2$ for some $0 < \theta < 1$ and $\mathbf{z}_1, \mathbf{z}_2 \in S$, it must hold that $\mathbf{y} = \mathbf{z}_1 = \mathbf{z}_2$.
- **Vertex:** Let $P \subseteq \mathbb{R}^n$ be a polyhedron. We say that $\mathbf{y} \in P$ is a *vertex* of $P$ if there is a direction $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{c}^\top\mathbf{y} < \mathbf{c}^\top\mathbf{z}$ for all $\mathbf{z} \in P \setminus \{\mathbf{y}\}$.




**Basic Feasible Solutions:**

- Assumptions ($\mathbf{A} \in \mathbb{R}^{m \times n}$):
  1. The polyhedron $P$ is not empty.
  2. The linear map $\mathbf{A}$ has full row rank.
- **Basis:** $m$ linearly dependent columns of $\mathbf{A}$.
- **Basic solution:** Let $\mathbf{B}$ be a basis. Then $\mathbf{x} =$
  - ▷ $\mathbf{B}^{-1}\mathbf{b}$ for the columns in basis
  - ▷ $\mathbf{0}$ for the columns not in basis

  is a basic solution.
- **Basic feasible solution:** Basic solution that $\geq \mathbf{0}$ (feasible).

## 3 Unimodularity and Total Unimodularity

**Polyhedron Integrality:** A polyhedron $P \subseteq \mathbb{R}^n$ is *integral* if all its extreme points are integer vectors.

- If so, then solving the relaxation of ILP is **tight**.

**Unimodularity (U):** We say that a square matrix $\mathbf{A} \in \mathbb{Z}^{m \times m}$ is *unimodular* if its determinant if its determinant is $\pm 1$. We say that a matrix $\mathbf{A} \in \mathbb{Z}^{m \times n}$ with **full row rank** is *unimodular* if the sub-matrix obtained by taking any $m$ columns of $A$ is either singular or unimodular (i.e., $\in \{-1, 0, 1\}$).

**Thm. 4.1** Let $\mathbf{A} \in \mathbb{Z}^{m \times n}$ be a matrix with **full row rank**. Then $\mathbf{A}$ is unimodular if and only if the set $P(\mathbf{b}) = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \geq \mathbf{0}\}$ is integral for any $\mathbf{b} \in \mathbb{Z}^m$ for which the polyhedron $P(\mathbf{b})$ is non-empty.

*Proof.* This can be proven by using **Cramer's Rule**.

- This applies only to **standard form**.
- **Prop. 4.2** A non-empty bounded polyhedron has at least one extreme point.
- **Prop. 4.3** Let $P$ be a non-empty polyhedron with at least one extreme point. The optimal solution of the LP $\{\min \mathbf{c}^\top\mathbf{x} : \mathbf{x} \in P\}$ is either $-\infty$ or is attained (possibly non-uniquely) at an extreme point of $P$.
- **Prop. 4.4** Any non-empty polyhedron of the form $\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}; \mathbf{x} \geq \mathbf{0}\}$ has at least one basic feasible solution (and hence an extreme point).

**Other Results:** (from tutorial)

- $\mathbf{U}$ is a square invertible matrix. $\mathbf{U}$ is unimodular if and only if $\mathbf{U}$ and $\mathbf{U}^{-1}$ are both integer-valued matrices (proven by Cramer's rule).
- $\mathbf{U}$ is a square invertible matrix. $\mathbf{U}$ is unimodular if and only if for all

$\mathbf{x}$, $\mathbf{Ux}$ is integral if and only if $\mathbf{x}$ is integral.

- **Unimodular operations:**
  1. Switch two columns;
  2. Multiply a column by $-1$;
  3. Add an integer multiple of a column to another;

  Let $\mathbf{U}$ be a square invertible matrix. $\mathbf{U}$ is unimodular if (and only if) it can be derived from the identity matrix via the above operations.

**Total Unimodularity (TU):** We say that a matrix $\mathbf{A} \in \mathbb{Z}^{m\times n}$ is *totally unimodular* if the determinant of each square sub-matrix of $\mathbf{A}$ is in $\{-1, 0, 1\}$.

**Thm. 4.6** Let $\mathbf{A} \in \mathbb{Z}^{m\times n}$ be a matrix. Then $\mathbf{A}$ is totally unimodular if and only if the set $P(\mathbf{b}) = \{\mathbf{x} : \mathbf{Ax} \le \mathbf{b}; \mathbf{x} \ge 0\}$ is integral for any $\mathbf{b} \in \mathbb{Z}^m$ for which the polyhedron $P(\mathbf{b})$ is non-empty.

*Proof.* $\mathbf{A}$ is TU

$\Leftrightarrow$[A I] is U (**Prop. 4.7**)

$\Leftrightarrow\{(\mathbf{x}, \mathbf{s}) : \mathbf{Ax} + \mathbf{s} = \mathbf{b}; \mathbf{x}, \mathbf{s} \ge 0\}$ is integral (**Thm. 4.1**)

$\Leftrightarrow\{\mathbf{x} : \mathbf{Ax} \le \mathbf{b}; \mathbf{x} \ge 0\}$ is integral (**Prop. 4.8**).

- **Prop. 4.7** $\mathbf{A}$ is TU if and only if $[\mathbf{A}\ \mathbf{I}_{m\times m}]$ is unimodular.
- **Prop. 4.8** $\mathbf{x}^*$ is an extreme point of $P(\mathbf{b}) = \{\mathbf{x} : \mathbf{Ax} \le \mathbf{b}; \mathbf{x} \ge 0\}$ if and only if $[\mathbf{x}^*\ \mathbf{s}^*] := [\mathbf{x}^*\ \mathbf{b} - \mathbf{Ax}^*]$ is an extreme point of $Q(\mathbf{b}) = \{(\mathbf{x}, \mathbf{s}) : \mathbf{Ax} + \mathbf{s} = \mathbf{b}; \mathbf{x}, \mathbf{s} \ge 0\}$.

**Prop. 4.9** $\mathbf{A} \in \mathbb{Z}^{m\times n}$. Then $\mathbf{A}$ is TU if and only if $\mathbf{A}^\top$ is TU.

**Prop. 4.10** $\mathbf{A} \in \mathbb{Z}^{m\times n}$. Then [A I], [A A], [A $-$A], [A $-$A I], [A $-$A I $-$I] are all TU. [I A]

**Thm. 4.11** Let $\mathbf{A} \in \mathbb{Z}^{m\times n}$ be a matrix. Then $\mathbf{A}$ is TU if and only if the set $\{\mathbf{x} : \mathbf{a} \le \mathbf{Ax} \le \mathbf{b}, \mathbf{l} \le \mathbf{x} \le \mathbf{u}\}$ is integral for all integral vectors $\mathbf{a}, \mathbf{b}, \mathbf{l}, \mathbf{u}$ for which the polyhedron is non-empty.

**Sufficient Conditions for TU:**

- **Thm. 4.12** A matrix $\mathbf{A} \in \{-1, 0, 1\}^{m\times n}$ is TU if both of the following conditions hold:
  1. Each column of $A$ contains at most 2 non-zero entries;
  2. It is possible to split the row indices $\{1, \cdots, m\}$ into two disjoint sets $I_1, I_2$ such that whenever a column (indexed by $j$) has 2 non-zero entries, it holds that
  $$\sum_{i\in I_1} A_{ij} = \sum_{i\in I_2} A_{ij}.$$
  In other words, if the two non-zeros have the same sign then one lies in $I_1$ and one lies in $I_2$, whereas if the two non-zeros have different signs then both lie in $I_1$ or both lie in $I_2$.
  - ▷ **Cor. 4.13** A matrix $\mathbf{A} \in \{-1, 0, 1\}^{m\times n}$ is TU if each of its columns has at most one $+1$ entry and at most one $-1$ entry.
- **Thm. 4.14** Let $\mathbf{A}$ be an $\times n$ integer-valued matrix, and for any $\mathcal{J} \subseteq \{1, \cdots, n\}$, let $\mathbf{A}_\mathcal{J}$ denote the $m \times \mathcal{J}$ sub-matrix obtained by keeping only the columns in $\mathcal{J}$. Then $\mathbf{A}$ is TU if and only if $\mathbf{A}_\mathcal{J}$ admits an equitable column-bicoloring for all non-empty $\mathcal{J} \subseteq \{1, \cdots, n\}$.
  - ▷ **Equitable bicoloring:** We say that an integer-valued matrix $\mathbf{A}$ admits an *equitable column-bicoloring* if it is possible to partition the columns indices $\mathcal{J}$ into two sets $\mathcal{J}_a$ and $\mathcal{J}_b$ so that the difference between the sums of the columns in these subsets is a vector with entries in $\{-1, 0, 1\}$:
  $$\sum_{i\in\mathcal{J}_a} A_i - \sum_{j\in\mathcal{J}_b} A_j \in \{-1, 0, 1\}^m,$$
  where $A_i$ is the $i$-th column of $\mathbf{A}$. Equivalently, there exists some $\mathbf{z} \in \{-1, +1\}^{|\mathcal{J}|}$ such that each entry of $\mathbf{Az}$ has absolute value at most one.
  - ▷ **Cor. 4.15** $\mathbf{A}$ is TU if and only if every sub-matrix obtained by taking a non-empty subset of the rows of $\mathbf{A}$ admits an equitable row-bicoloring.
- **Prop. 4.16** The node-edge incidence matrix of an undirected bipartite graph is TU.
  - ▷ $A_{ij} = 1$ if node $i$ is in edge $j$ and 0 otherwise.
  - ▷ **Graph matching:** Given a bipartite graph, a *matching* is a subset of non-intersecting edges (i.e., edges for which no two of them have a common node). A matching is said to be *perfect* if all nodes are selected.
- **Prop. 4.17** The node-edge incidence matrix of a directed graph is TU.
  - ▷ $A_{ij} = 1$ if edge $j$ starts from node $i$, $-1$ if edge $j$ ends at node $i$, 0 otherwise.

## 4 Appendix: Mathematical Facts

**Determinant:**

- Cofactor expansion: For a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, we have
$$\det(\mathbf{A}) = \sum_{j=1}^{n} A_{ij}C_{ij},$$
where $C_{ij} = (-1)^{i+j}\det(\mathbf{A}_{ij})$.
- **Cramer's Rule** $\mathbf{A} \in \mathbb{R}^{n\times n}$ is invertible. Then the solution to the linear system $\mathbf{Ax} = \mathbf{b}$ is given by
$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})},$$
where $\mathbf{A}_i$ is the matrix obtained by replacing the $i$-th column of $\mathbf{A}$ with vector $\mathbf{b}$:
$$[A_1 \quad \cdots \quad A_{i-1} \quad \mathbf{b} \quad A_{i+1} \quad \cdots \quad A_n].$$
- $\det(\mathbf{A}^\top) = \det(\mathbf{A})$.
- $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$.
- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

**Trace:**

- $\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$.
- $\mathrm{tr}(\mathbf{A}^\top) = \mathrm{tr}(\mathbf{A})$.
- $\mathrm{tr}(\mathbf{A}^\top B) = \mathrm{tr}(\mathbf{AB}^\top) = \mathrm{tr}(\mathbf{B}^\top\mathbf{A}) = \mathrm{tr}(\mathbf{BA}^\top)$.

(P) $\min\ c^T x$

s.t. $a_i^T x \ge b_i \quad \forall i \in M^+;$
$a_i^T x \le b_i \quad \forall i \in M^-;$
$a_i^T x = b_i \quad \forall i \in M_0;$
$x_j \ge 0 \quad \forall j \in N^+;$
$x_j \le 0 \quad \forall j \in N^-;$
$x_j \in \mathbb{R} \quad \forall j \in N_{\mathbb{R}};$

(D) $\max\ p^T b$

s.t. $p_i \ge 0 \quad \forall i \in M^+;$
$p_i \le 0 \quad \forall i \in M^-;$
$p_i$ free $\quad \forall i \in M_0;$
$p^T A_j \le c_j \quad \forall j \in N_+$
$p^T A_j \ge c_j \quad \forall j \in N_-$
$p^T A_j = c_j \quad \forall j \in N_{\mathbb{R}}$

Weak duality theorem: If $x$ is feasible in (P) and $p$ is feasible in (D), then $p^T b \le c^T x$ and thus $\sup_{p \text{ feasible}} p^T b \le \inf_{x \text{ feasible}} c^T x$.

- If feasible and $p^T b = c^T x$, both $p$ and $x$ are optimal.
- Unboundness in one implies infeasibility in another.

(P) and (D) can both be infeasible.

Maximum flow problem:

$$\max_{\{f_{uv}\}_{(u,v)\in E}} \sum_{v:(s,v)\in E} f_{sv}$$

s.t. $0 \le f_{uv} \le C_{uv} \quad \forall (u,v) \in E$

$$\sum_{u:(u,v)\in E} f_{uv} = \sum_{w:(v,w)\in E} f_{vw} \quad \forall v \in (V\setminus\{s,t\})$$

lagrangian dual

Minimum cut problem:

$$\min_{\substack{\{\lambda_{uv}\}_{(u,v)\in E} \\ \{\mu_v\}_{v\in V}}} \sum_{(u,v)\in E} C_{uv}\lambda_{uv}$$

$\mu, \lambda \in \{0,1\}$.

s.t. $\mu_s = 1, \mu_t = 0, \quad \lambda_{uv} \ge \mu_u - \mu_v$, $\lambda_{uv} \ge 0 \quad \forall (u,v) \in E$.

Shortest path problem: $s \to v_1 \to v_2 \to \cdots \to v_k \to t$

$G = (V, E)$ is a directed graph. $l_e$ is length of $e$.

$$\min_{\{x_e\}} \sum_{e\in E} l_e x_e$$

s.t. $\sum_{e\in\delta^-(v)} x_e - \sum_{e\in\delta^+(v)} x_e = 0 \quad \forall v \in V\setminus\{s,t\}$

$\sum_{e\in\delta^-(s)} x_e - \sum_{e\in\delta^+(s)} x_e = -1$

$\sum_{e\in\delta^-(t)} x_e - \sum_{e\in\delta^+(t)} x_e = 1$

$x \in \{0,1\}^E$.

LP relaxation is tight.