MA4264 Game Theory
Final Examination Helpsheet
AY2023/24 Semester 2 · Prepared by Tian Xiao *@snoidetx*

1 Static Games of Complete Information

Representation (Normal-Form)

$$G = \{S_1, S_2, \cdots, S_n; u_1, u_2, \cdots, u_n\}.$$

Payoff functions: $u_1(s_{1,1}, s_{2,1}) = 1$, $u_2(s_{1,1}, s_{2,1}) = 0$

		Player 2 (Strategy spaces / Strategies)		
		$s_{2,1} \in \boxed{S_2}$	$s_{2,2} \in S_2$	$\boxed{s_{2,3}} \in S_2$
Player 1	$s_{1,1} \in S_1$	$\boxed{1, 0}$	1, 2	0, 1
	$s_{1,2} \in S_1$	0, 3	0, 1	2, 0

Strategy

Domination: In $G = \{S_1, S_2, \cdots, S_n; u_1, u_2, \cdots, u_n\}$, let $s_i', s_i'' \in S_i$. Strategy s_i' is *strictly dominated* by strategy s_i'' if

$$\forall s_{-i} \in S_{-i}\ [u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i})],$$

where $-i$ represents the set of other players.

- Rational players do not play strictly dominated strategies.
- Iterative elimination of strictly dominated strategies (IESDS):

① Right dominated by Middle.
② Down dominated by Up.
③ Left dominated by Middle.

		Player 2		
		③ Left	Middle	① Right
Player 1	② Up	1, 0	1, 2	0, 1
	Down	0, 3	0, 1	2, 0

Best response: Given strategies s_{-i} of other players, the *best response* of player i is

$$R_i(s_{-i}) = \max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

- Check ① $\frac{d}{ds_i} u_i(s_i, s_{-i}) = 0$ & $\frac{d^2}{ds_i^2} u_i(s_i, s_{-i}) < 0$ and ② **boundaries**.
- Response curve: Graph of $R_i(s_{-i})$ against s_{-i}.

Nash equilibrium $(s_1^*, s_2^*, \cdots, s_n^*)$: $\forall i = 1, 2, \cdots, n\ [s_i^* \in R_i(s_{-i}^*)]$.

- No player has incentives to deviate from Nash equilibrium.
- **Prop 1.** {Nash equilibrium} $\subseteq$ {IESDS}.
- **Prop 2.** In a game G with finite $S_1, S_2, \cdots, S_n$, if {IESDS} contains only one $(s_1^*, s_2^*, \cdots, s_n^*)$, then it is the unique Nash equilibrium.
- Nash equilibrium is the **intersection** of all response curves.

Mixed strategy: A probability distribution

$$p_i = (p_{i1}, p_{i2}, \cdots, p_{iK}), \text{ where } \sum_{k=1}^{K} p_{ik} = 1 \text{ and } p_{ik} \geq 0,$$

w.r.t. each pure strategy $s_{ik} \in S_i$.

- Expected payoff (2-player): $v_1(p_1, p_2) = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k})$.

Nash equilibrium: Each player's mixed strategy is a best response to the other player's mixed strategy:

$$v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*); \quad v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2).$$

- **Thm 1.** If n is finite and S_i is finite for every i, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

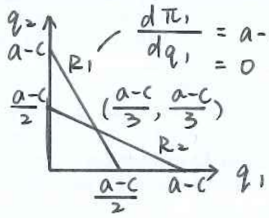
<u>Cournot Model of Duopoly</u>

profit $\pi_i = Pq_i - cq_i$

$P = \begin{cases} a - Q & \text{if } Q < a \\ 0 & \text{if } Q \geq a \end{cases}$

$Q = q_1 + q_2$

$\frac{d\pi_1}{dq_1} = a - q_2 - 2q_1 - c = 0$



n firm: all $\frac{a-c}{n+1}$.

<u>Bertrand Model of Duopoly</u>

Choose p_1 and p_2. $q_i(p_i, p_j) = a - p_i + bp_j$ $(b < 2)$

$\left.\begin{array}{l} p_1^* = \frac{a+c}{2-b} \\ p_2^* = \frac{a+c}{2-b} \end{array}\right\}$

<u>Infinitely Repeated Game</u>

Discount factor $\delta \in (0, 1)$

		Player 2	
		L_2	R_2
Player 1	L_1	1, 1	5, 0
	R_1	0, 5	4, 4

General case: x, x $\quad z, 0$
$0 < x < y < z$ $\quad 0, z$ $\quad y, y$

Trigger is optimal iff $\delta \geq \frac{z-y}{z-x}$.

Non-cooperative strategy:
Always play (L_1, L_2)
$\pi_1 = 1 + \delta + \delta^2 + \cdots = \frac{1}{1-\delta} = \pi_2$

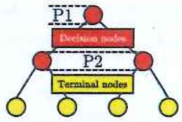
Trigger strategy:
Play (R_1, R_2) until someone changes
$\pi_1 = 4 + 4\delta + \cdots = \frac{4}{1-\delta} = \pi_2$

if player 1 changes,
$\pi_1 = 5 + \delta + \delta^2 + \cdots = 5 + \frac{\delta}{1-\delta}$
When $\frac{4}{1-\delta} > 5 + \frac{\delta}{1-\delta}$, i.e., $\delta \geq \frac{1}{4}$,
trigger strategy is a Nash equilibrium.

2 Dynamic Games of Complete Information

Representation (Extensive-Form)

① Players in the game; AND
② When each player has the move; AND
③ What each player can do at each move; AND
④ What each player knows at each move; AND
⑤ The payoffs for each combination of moves.



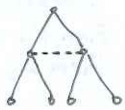
- Payoff functions: $u_i(a_1, a_2, \cdots, a_m)$, where $a_1, a_2, \cdots, a_m$ are a sequence of actions.
  - ▷ Complete information: Payoffs are common knowledge.
- Can be transformed into **normal-form** by specifying payoffs for each combination of strategies.
- **Subgame:**
  ① Begin at a singtelton information set (not the root); AND
  ② Include the whole remaining subtree; AND
  ③ Do not cut any information set.

Information Set for a Player

A collection of decision nodes s.t.

① the player needs to move at every node in the information set; AND
② when the play of the game reaches a node in the information set, the player with the move does not know which node in the set has (or has not) been reached.
  ▷ The set of feasible actions at each decision node must be same.

- Perfect information: All previous moves are observed before next move is chosen.
- Imperfect information: Some information sets are non-singleton.
  - ▷ Decision nodes in an information set connected by a dotted line.

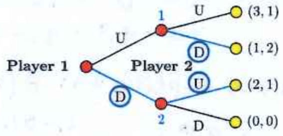


Strategy

A complete plan of actions $s = (s_1, s_2, \cdots, s_n)$.

- Actions specified by s: $(a_1(s), a_2(s), \cdots, a_m(s))$.
- Payoff received by playing s: $\tilde{u} = u(a_1(s), a_2(s), \cdots, a_m(s))$.

Backwards Induction



subgame-perfect outcome: (D, U)
subgame-perfect NE: (D, DU)

Nash Equilibrium

Obtained from the normal-form representation of strategies.

- **Subgame-perfect Nash equilibrium**: A Nash equilibrium is *subgame-perfect* if the players' strategies constitute a Nash equilibrium in every subgame.
  - ▷ Not all Nash equilibria are subgame-perfect.

<u>Stackelberg Model of Duopoly</u>

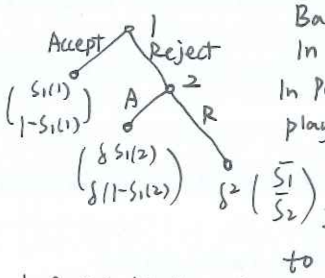
① Firm 1 chooses $q_1 \geq 0$.
② Firm 2 chooses $q_2 \geq 0$.

$Q = q_1 + q_2$,
$\pi_i(q_1, q_2) = q_i[P(Q) - c]$,
$P(Q) = \begin{cases} a - Q, & \text{if } Q < a \\ 0, & \text{if } Q \geq a. \end{cases}$

BR of firm 2: $R_2(q_1) = \begin{cases} \frac{a - q_1 - c}{2} & \text{if } q_1 < a - c \\ 0 & \text{otherwise} \end{cases}$

Firm 1 knows $R_2(q_1)$,
so it will maximize $\pi_1(q_1, R_2(q_1))$. $\Rightarrow q_1^* = \frac{a-c}{2}$, $q_2^* = \frac{a-c}{4}$.
$\underbrace{}_{\text{subgame-perfect outcome}}$
$(s_1^* = \frac{a-c}{2}, R_2(\cdot))$ is the (subgame-perfect) Nash equilibrium.

<u>Sequential-Bargaining Game</u>



Backward induction:
In Period 3 Player 1 receives $\delta^2 \bar{s}_1$.
In Period 2 Player 2 offers $\delta \bar{s}_1$ to player 1 and $1 - \delta \bar{s}_1$ to himself.
So he will receive $\delta(1 - \delta \bar{s}_1)$.
In period 1 player 1 offers $\delta(1 - \delta \bar{s}_1)$ to player 2 and $1 - \delta(1 - \delta \bar{s}_1)$ to himself.

<u>Infinite Horizon Bargaining Game</u>

Let $(\bar{u}_1, \bar{u}_2)$ be the optimal payoff. Players can regard these payoffs as a settlement in Period 3, that is,
$\bar{u}_1 = 1 - \delta(1 - \delta \bar{u}_1) \Rightarrow \bar{u}_1 = \frac{1}{1+\delta}$, $\bar{u}_2 = \frac{\delta}{1+\delta}$.

3 Static Games of Incomplete Information

Representation (Normal-Form)

$$G = \{A_1, \cdots, A_n; T_1, \cdots, T_n; P_1, \cdots, P_n; u_1, \cdots, u_n\}, \text{ where}$$

- $A_1, \cdots, A_n$ are each player's *action space*;
- $T_1, \cdots, T_n$ are each player's *type space*;
  - ▷ Player i knows his own type t_i.
- $P_1, \cdots, P_n$ are each player's *belief*;
  - ▷ Player i only knows a distribution of other players' types $P_i(t_{-i}|t_i)$.
  - ▷ Bayes rule: $P_i(t_{-i}|t_i) = \frac{P(t_{-i},t_i)}{\sum_{t'_{-i} \in T_{-i}} P(t'_{-i},t_i)}$.
- $u_1, \cdots, u_n$ are each player's *payoff function*.
  - ▷ Player i's payoff is $u_i(a_1, \cdots, a_n; t_i)$.

Strategy

$$s_i : T_i \to A_i, \text{ where } s_i(t_i) \text{ gives the action.}$$

Bayesian Nash Equilibrium

$$\begin{aligned} s_i^*(t_i) &= \arg\max_{a_i \in A_i} \mathbb{E}_{t_{-i}} \left[u_i(a_i, s^*_{-i}(t_{-i}); t_i)\right] \\ &= \arg\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} P_i(t_{-i}|t_i) u_i(a_i, s^*_{-i}(t_{-i}); t_i). \end{aligned}$$

- **Prop. 1.** $(s_1^*, \cdots, s_n^*)$ is a Bayesian Nash equilibrium, if $\forall t_i \in T_i, a_i \in A_i, a_{-i} \in A_{-i}\ [u_i(s_i^*(t_i), a_{-i}; t_i) \geq u_i(a_i, a_{-i}; t_i)]$.

Cournot competition under Asymmetric Information

Firm 1's cost function $C_1(q_1) = cq_1$

Firm 2's cost function $C_2(q_2) = \begin{cases} c_H q_2 & \text{with prob } \theta \\ c_L q_2 & \text{with prob } 1-\theta \end{cases}$

Firm 2 maximizes $(a - q_1^* - q_2 - c_H) q_2$ if c_H,
$(a - q_1^* - q_2 - c_L) q_2$ if c_L.

Firm 1 maximizes expectation: $\theta(a - q_1 - q_{2H} - c) q_1 + (1-\theta)(a - q_1 - q_{2L} - c) q_1$.

We get $q_{2H} = \frac{a - q_1^* - c_H}{2}$, $q_{2L} = \frac{a - q_1^* - c_L}{2}$

$q_1^* = \frac{\theta(a - q_{2H}^* - c) + (1-\theta)(a - q_{2L}^* - c)}{2}$ } solve.

Providing public goods under incomplete information

		P_2 contribute	P_2 don't
P_1	contribute	$1-c_1, 1-c_2$	$1-c_1, 1$
	don't	$1, 1-c_2$	$0, 0$

$c_1 = \begin{cases} 0.5 & \text{prob} = 0.5 \\ 1.2 & \text{prob} = 0.5 \end{cases}$

$c_2 = 0.8$ prob $= 1$

corresponding to its own type

Type of player 1: $\{\{0.5\}, \{1.2\}\} \Rightarrow \{CD, CC, DC, DD\}$

Type of player 2: $\{0.8\}$. $\Rightarrow \{C, D\}$.

player 1's br: $R_1(C) = DD$, $R_1(D) = CD$

player 2's br: $R_2(CC) = D$, $R_2(CD) = D$, $R_2(DC) = D$, $R_2(DD) = C$

NEs: (DD, C), (CD, D).

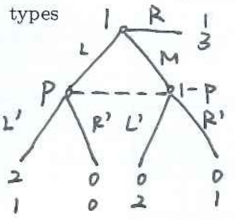
4 Dynamic Games of Incomplete Information

Strategy

Perfect Bayesian Nash Equilibrium

① **Consistency:** At each information set, beliefs are determined by Bayes rule and the players' strategies, wherever possible.

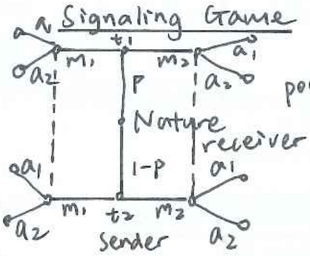
② **Sequentially rational:** at each information set, given a player's belief, its action must be optimal.



Two NE: (R, R') and (L, L').

- If (R, R'), the information set is not reached
  expectation when playing $L' = p + 2(1-p) = 2-p$
  expectation when playing $R' = 1-p < 2-p$.
  not sequentially optimal. Not PBNE.
- if (L, L'), $p = 1$, L' is optimal. PBNE.

Signaling Game



Sender's strategy: (2 types) separating

pooling $\{m_1m_1, m_1m_2, m_2m_1, m_1m_2\}$

Receiver's strategy: (seeing m_1/m_2)

$\{a_1a_1, a_1a_2, a_2a_1, a_2a_2\}$

Consistency: If m_1m_1, then receiver's belief at the left $= (p, 1-p)$, right is arbitrary.

If m_1m_2, both are $(1,0)$ and $(0,1)$.

Nature Selects a Game

Nature selects from {Game 1, Game 2, Game 3} with prob $= \frac{1}{3}$. Player 1 learns whether nature has selected game 1; Player 2 does not learn.

player 1's types: $\{\{1\}, \{2,3\}\} \Rightarrow \{TT, TB, BT, BB\}$

player 2's types: $\{\{1,2,3\}\} \Rightarrow \{L, R\}$.

player 1's br and player 2's br.

First-price Sealed bid auction $v_1, v_2 \sim U(0,1)$.

b_1, b_2 higher win. draw flip coin.

Formulation:

$A_1 = A_2 = [0, \infty)$ (b)

$T_1 = T_2 = [0, 1]$ (v)

payoff: $v_i - b_i$ if $b_i > b_j$; $\frac{v_i - b_i}{2}$ if draw; 0 if $b_i < b_j$.

The unique symmetric Bayesian NE is $b_1(v_1) = v_1/2$ and $b_2(v_2) = v_2/2$.

Cooperative Games

Representation

Two-Person Bargaining Game: The pair $\Gamma = (H, d)$ is a **two-person bargaining game** if (1) $H \subset \mathbb{R}^2$ is compact and convex; (2) $d \in H$; (3) H contains at least one element u such that $u >> d$. The set of two-person bargaining games is denoted W.

n-Person Game: For an n-person game with the set of players $N = \{1, 2, ..., n\}$, any non-empty subset of N is called a **coalition**. For each coalition S, the **characteristic function** v of the game gives the amount $v(S)$ that the coalition can be sure of receiving. The game is $\Gamma(N, v)$.

① $v(\emptyset) = 0$.

② Super-aditivity: for any disjoint coalitions $K, L \subseteq N$, $v(K \cup L) \geq v(K) + v(L)$.

Strategy

Domination: Let (u, v) and (u', v') be two payoff pairs. We say (u, v) **dominates** (u', v') if

$$u \geq u', \ v \geq v'.$$

Payoff pairs not dominated by any other are **Pareto-optimal**.

Nash Bargaining Solution: A mapping $f : W \to \mathbb{R}^2$ that associates a unique element $f(H, d) = (f_1(H, d), f_2(H, d))$ with the game $(H, d) \in W$, satisfying the following axioms:

① *Feasibility*: $f(H, d) \in H$.

② *Individual Rationality*: $f(H, d) \geq d$ for all $(H, d) \in W$.

③ *Pareto Optimality*: $f(H, d)$ is Pareto optimal.

④ *Invariance under Linear Transformations*: Let $a_1, a_2 > 0, b_1, b_2 \in \mathbb{R}$, and $(H, d), (H', d') \in W$ where $d'_i = a_i d_i + b_i$, and $H' = \{x \in \mathbb{R}^2 | x_i = a_i y_i + b_i, y \in H\}$ $(i = 1, 2)$. Then $f_i(H'_i, d'_i) = a_i f_i(H, d) + b_i$.

⑤ *Symmetry*: If $d_1 = d_2$ and $(x_1, x_2) \in H \to (x_2, x_1) \in H$, then $f_1(H, d) = f_2(H, d)$.

⑥ *Independence of Irrelevant Alternatives*: If $(H, d), (H', d') \in W$ and $d = d', H \subset H'$ and $f(H', d') \in H$, then $f(H, d) = f(H', d')$.

- A game (H, d) has a unique Nash solution $u^* = f(H, d)$ satisfying conditions $1-6$ if and only if
$$(u_1^* - d_1)(u_2^* - d_2) > (u_1 - d_1)(u_2 - d_2)$$
for all $u \in H, u \geq d$ and $u \neq u^*$.

Imputation: An **imputation** in the game (N, v) is a payoff vector $x = (x_1, ..., x_n)$ satisfying

① *Group rational:* $\sum_{i=1}^n x_i = v(N)$.

② *Individually rational:* $x_i \geq v(\{i\})$ for all $i \in N$.

Let $I(N, v)$ denote the set of all imputations.

Imputation Domination: Let $x, y \in I(N, v)$, and let S be a coalition. We say x **dominates** y **via** S ($x \succ_S y$) if (1) $x_i > y_i$ for all $i \in S$; (2) $\sum_{i \in S} x_i \leq v(S)$. We say x **dominates** y if there exists any S such that $x \succ_S y$.

Core: The set of all undominated imputations for a game (N, v) is called the **core**, denoted by $C(N, v)$.

- The core of the game is the set of all n-vectors, satisfying
  ① $\sum_{i \in S} x_i \geq v(S)$ for all $\emptyset \neq S \subset N$.
  ② $\sum_{i \in N} x_i = v(N)$.

4.1 Shapley Value

The **Shapley value** is an n-vector, denoted by $\phi(v)$, satisfying a set of axioms. The i-th component of $\phi(v)$ can be uniquely determined as

$$\begin{aligned} \phi_i(v) &= \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)] \\ &= \frac{1}{n} \sum_{s=0}^{n-1} \frac{1}{\binom{n-1}{s}} \sum_{S \subseteq N \setminus \{i\}, |S| = s} [v(S \cup \{i\}) - v(S)]. \end{aligned} \quad (1)$$

- The Shapley value has individual rationality, efficiency, symmetry, additivity, dummy properties.