MA4264 Game Theory

Midterm Examination Helpsheet

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1 Static Games of Complete Information

n-person game (normal-form): $G = \{S_1, S_2, \cdots, S_n; u_1, u_2, \cdots, u_n\}$.

Payoff functions: $u_1(s_{1,1}, s_{2,1}) = 1$, $u_2(s_{1,1}, s_{2,1}) = 0$

Strategy spaces · Strategies

		Player 2		
		$s_{2,1} \in S_2$	$s_{2,2} \in S_2$	$s_{2,3} \in S_2$
Player 1	$s_{1,1} \in S_1$	1, 0	1, 2	0, 1
	$s_{1,2} \in S_1$	0, 3	0, 1	2, 0

1.1 Pure Strategy Games

Domination: In $G = \{S_1, S_2, \cdots, S_n; u_1, u_2, \cdots, u_n\}$, let $s_i', s_i'' \in S_i$. Strategy s_i' is *strictly dominated* by strategy s_i'' if

$$\forall s_{-i} \in S_{-i} \left[u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i})\right],$$

where $-i$ represents the set of other players.

- Rational players do not play strictly dominated strategies.
- Iterative elimination of strictly dominated strategies (IESDS):

① Right dominated by Middle.
② Down dominated by Up.
③ Left dominated by Middle.

		Player 2		
		③ Left	Middle	① Right
Player 1	Up	1, 0	1, 2	0, 1
	② Down	0, 3	0, 1	2, 0

Best response: Given strategies s_{-i} of other players, the *best response* of player i is

$$R_i(s_{-i}) = \max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

- Check ① $\frac{\mathrm{d}}{\mathrm{d}s_i} u_i(s_i, s_{-i}) = 0$ & $\frac{\mathrm{d}^2}{\mathrm{d}s_i^2} u_i(s_i, s_{-i}) < 0$ and ② **boundaries**.
- Response curve: Graph of $R_i(s_{-i})$ against s_{-i}.

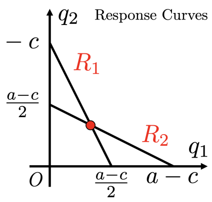
Nash equilibrium $(s_1^*, s_2^*, \cdots, s_n^*)$**:** $\forall i = 1, 2, \cdots, n\ [s_i^* \in R_i(s_{-i}^*)]$.

- No player has incentives to deviate from Nash equilibrium.
- **Prop 1.** {Nash equilibrium} $\subseteq$ {IESDS}.
- **Prop 2.** In a game G with finite $S_1, S_2, \cdots, S_n$, if {IESDS} contains only one $(s_1^*, s_2^*, \cdots, s_n^*)$, then it is the unique Nash equilibrium.
- Nash equilibrium is the **intersection** of all response curves.

Example. Consider the *Cournot model of duopoly* where two firms sell the same product.

- ▷ quantity q_1, q_2, cost c, fixed cost c_0;
- ▷ price $p = \begin{cases} a - (q_1 + q_2) & \text{if } Q < a \\ 0 & \text{if } Q \geq a \end{cases}$;
- ▷ profit $\pi = pq - cq - c_0$.

The Nash equilibrium is $q_1^* = q_2^* = \frac{1}{3}(a - c)$.



1.2 Mixed Strategy Games

Mixed strategy: A probability distribution

$$p_i = (p_{i1}, p_{i2}, \cdots, p_{iK}), \text{where } \sum_{k=1}^{K} p_{ik} = 1 \text{ and } p_{ik} \geq 0,$$

w.r.t. each pure strategy $s_{ik} \in S_i$.

- Expected payoff (2-player): $v_1(p_1, p_2) = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} u_1(s_{1j}, s_{2k})$.

Nash equilibrium: Each player's mixed strategy is a best response to the other player's mixed strategy:

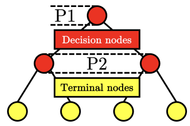
$$v_1(p_1^*, p_2^*) \geq v_1(p_1, p_2^*); \quad v_2(p_1^*, p_2^*) \geq v_2(p_1^*, p_2).$$

- **Thm 1.** If n is finite and S_i is finite for every i, then there exists at least one Nash equilibrium, possibly involving mixed strategies.

2 Dynamic Games of Complete Information

n-person game (extensive-form): The following are specified:

① the players in the game; AND
② when each player has the move; AND
③ what each player can do at each move; AND
④ what each player knows at each move; AND
⑤ the payoffs for each combination of moves.



- Payoff functions: $u_i(a_1, a_2, \cdots, a_m)$, where $a_1, a_2, \cdots, a_m$ are a sequence of actions.
  - ▷ Complete information: Payoffs are common knowledge.
- Strategy: A complete plan of actions $s = (s_1, s_2, \cdots, s_n)$.
  - ▷ Actions specified by s: $(a_1(s), a_2(s), \cdots, a_m(s))$.
  - ▷ Payoff received by playing s: $\tilde{u} = u(a_1(s), a_2(s), \cdots, a_m(s))$.
- Nash equilibrium: Obtained from the normal-form representation of **strategies**.

Information set for a player: A collection of decision nodes s.t.

① the player needs to move at every node in the information set; AND
② when the play of the game reaches a node in the information set, the player with the move does not know which node in the set has (or has not) been reached.
  ▷ The set of feasible actions at each decision node must be same.

- Perfect information: All previous moves are observed before next move is chosen.
- Imperfect information: Some information sets are non-singleton.

Backwards induction: