ST2334 Probability and Statistics

Midterm Examination Helpsheet

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1 Basic Concepts of Probability

Terminology	Definition	Example
Statistical experiment	Procedure that produces data or observations	Rolling a dice
Sample space S	Set of all possible outcomes of a statistical experiment	$\{1, 2, 3, 4, 5, 6\}$
Sample point	An outcome in the sample space	1
Event	A subset of the sample space	An odd number facing up

- The sample space itself is an event and called a *sure event*.
- An event that contains no element is called a *null event* $\emptyset$.

Event: union $A \cup B$, intersection $A \cap B$, complement A'.

- Mutually exclusive/disjoint: $A \cap B = \emptyset$.
  - ▷ $P(A \cap B) = 0$ does not mean mutually exclusive (e.g., continuous).
- Contained: $A \subset B$.
- Equivalent: $A \subset B$ and $B \subset A \Leftrightarrow A = B$.

$A \cap A' = \emptyset$	$A \cap \emptyset = \emptyset$	$A \cup A' = S$	$(A')' = A$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup B = A \cup (B \cap A')$	$A = (A \cap B) \cup (A \cap B')$

- De Morgan's law: $(A_1 \cup A_2 \cup \cdots \cup A_n)' = A_1' \cap A_2' \cap \cdots \cap A_n'$; $(A_1 \cap A_2 \cap \cdots \cap A_n)' = A_1' \cup A_2' \cup \cdots \cup A_n'$.

Counting: Multiplication principle + Addition principle.

- Permutation: $P_r^n = \frac{n!}{r!(n-r)!}$.
- Combination: $C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Probability: How likely event A occurs, $P(A)$.

- Relative frequency: $f_A = \frac{n_A}{n} \to P(A)$ as $n \to \infty$.
  - ▷ $0 \le f_A \le 1$;
  - ▷ $f_A = 1$ if A occurs in every repetition;
  - ▷ If A and B are mutually exclusive, then $f_{A \cup B} = f_A + f_B$.
- Axioms of probability:
  - ① For any event A, $0 \le P(A) \le 1$;
  - ② $P(S) = 1$;
  - ③ $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$.
  - ▷ $P(\emptyset) = 0$;
  - ▷ If $A_1, A_2, \cdots, A_n$ are mutually exclusive events, then $P_{A_1 \cup A_2 \cup \cdots A_n} = P(A_1) + P(A_2) + \cdots + P(A_n)$.
  - ▷ For any event A, $P(A') = 1 - P(A)$.
  - ▷ For any events A, B, $P(A) = P(A \cap B) + P(A \cap B')$.
  - ▷ For any events A, B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
  - ▷ If $A \subset B$, then $P(A) \le P(B)$.
- Conditional probability: $P(B|A) = \frac{P(A \cap B)}{P(A)}$.
  - ▷ $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$ if $P(A), P(B) \ne 0$.
  - ▷ $P(A|B) = \frac{P(A)P(B|A)}{P(B)}$.
- Independence: A and B are *independent* if and only if $P(A \cap B) = P(A)P(B)$. We denote this by $A \perp B$.
  - ▷ If $P(A) \ne 0$, $A \perp B$ if and only if $P(B|A) = P(B)$.
- Law of total probability: Suppose $A_1, A_2, \cdots, A_n$ is a partition of S. Then, $P(B) = \sum_{i=1}^n P(B \cap A_i) = \sum_{i=1}^n P(A_i)P(B|A_i)$.
- Bayes' theorem: Suppose $A_1, A_2, \cdots, A_n$ is a partition of S. Then, $P(A_k|B) = \frac{P(A_k)P(B|A_k)}{\sum_{i=1}^n P(A_i)P(B|A_i)}$.

2 Random Variables

Random variable: A random variable $X : S \to \mathbb{R}$ assigns a real number to every $s \in S$.

- Range space: $R_X = \{x \mid x = X(s), s \in S\}$. Either finite or countable.
- Discrete r.v.: $R_X = \{x_1, x_2, x_3, \cdots\}$.
  - ▷ Probability (mass) function: $f(x) = \begin{cases} P(X = x) & \text{for } x \in R_X; \\ 0 & \text{for } x \notin R_X. \end{cases}$
  - ▷ Probability distribution: Collection of pairs $(x_i, f(x_i))$.
  - ▷ Properties of p.m.f.:
    - ① $f(x_i) \ge 0$ for all $x_i \in R_X$;
    - ② $f(x_i) = 0$ for all $x_i \notin R_X$;
    - ③ $\sum_{x_i \in R_X} f(x_i) = 1$.
    - ④ For any $B \subset \mathbb{R}$, $P(X \in B) = \sum_{x_i \in B \cap R_X} f(x_i)$.
- Continuous r.v.:
  - ▷ Probability density function:
    - ① $f(x) \ge 0$ for all $x \in R_X$; and $f(x) = 0$ for all $x \notin R_X$;
    - ② $\int_{R_X} f(x)\,\mathrm{d}x = 1$;
    - ③ $P(a \le X \le b) = \int_a^b f(x)\,\mathrm{d}x$.
- Cumulative distribution function: $F(x) = P(X \le x)$.
  - ▷ $P(a \le X \le b) = P(X \le b) - P(X < a) = F(b) - F(a-)$.
  - ▷ Continuous: $P(a \le X \le b) = P(a < X < b) = F(b) - F(a)$.
  - ▷ Right continuous: $F(a) = \lim_{x \to a^+} F(x)$.
  - ▷ Convergence to 0 and 1 in the limits: $\lim_{x \to -\infty} F(x) = 0$; $\lim_{x \to +\infty} F(x) = 1$.

Expectation:

- Discrete: $\mathbb{E}[X] = \mu_X = \sum_{x_i \in R_X} x_i f(x_i)$.
- Continuous: $\mathbb{E}[X] = \mu_X = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x = \int_{x \in R_X} x f(x)\,\mathrm{d}x$.
- $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$.
- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
- Discrete: $\mathbb{E}[g(X)] = \sum_{x \in R_X} g(x)f(x)$; continuous: $\mathbb{E}[g(X)] = \int_{R_X} g(x)f(x)\,\mathrm{d}x$.

Variance: $\sigma_X^2 = V(X) = \mathbb{E}[(X - \mu_X)^2]$.

- Discrete: $V(X) = \sum_{x \in R_X} (x - \mu_X)^2 f(x)$;
- Continuous: $V(X) = \mathrm{d}_{R_X} (x - \mu_X)^2 f(x)\,\mathrm{d}x$.
- $V(aX + b) = a^2 V(X)$.
- $V(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.
- Standard deviation: $\sigma_X = \sqrt{V(X)}$.

3 Special Probability Distributions

Discrete Distributions

Discrete uniform distribution

p.m.f.: $f_X(x) = \begin{cases} \frac{1}{k}, & x = x_1, x_2, \dots, x_k; \\ 0, & \text{otherwise}. \end{cases}$

$\mathbb{E}[X] = \frac{1}{k}\sum_{i=1}^{k} x_i \quad V[X] = \sigma_x^2 = \frac{1}{k}\sum_{i=1}^{k} x_i^2 - \mu_x^2.$

Bernoulli Distribution: $X \sim \text{Bern}(p), 0 \le p \le 1$.

p.m.f.: $f_X(x) = P(X = x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \end{cases} = p^x(1-p)^{1-x}.$

$\mu_X = \mathbb{E}[X] = p$

$\sigma^2_X = V(X) = p(1-p) = pq$

Binomial Distribution: No. of successes in n trails of Bernoulli processes; $X \sim \text{Bin}(\text{n}, \text{p})$.

$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x = 0, 1, \cdots, n$.

$\mu_X = \mathbb{E}[X] = np; \sigma_X^2 = V(X) = np(1-p)$.

Negative binomial distribution: No. of i.i.d. Bernoulli trials until first k successes.

$f_X(x) = P(X = x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$ for $x = k, k+1, \dots$.

$\mathbb{E}[X] = \frac{k}{p}$. $V(X) = \frac{(1-p)k}{p^2}$.

Geometric distribution: No. of i.i.d. Bernoulli trials untail first success.

$f_X(x) = P(X = x) = p(1-p)^{x-1}$.

$\mathbb{E}[X] = \frac{1}{p} \quad V(X) = \frac{1-p}{p^2}$.

Poisson distribution: No. of events occuring in a fixed period of time/fixed region; $X \sim \text{Poisson}(\lambda)$.

$\mathbb{E}[X] = \lambda \quad V(X) = \lambda$.

Poisson process: continous-time process with rate α. Poisson(αT).

Poisson approximation to Binomial: Let $X \sim (n, p)$.
Suppose $n \to \infty$, $p \to 0$ s.t. np is constant, $\lim_{p \to 0, n \to \infty} P(X=x) = \frac{e^{-np}(np)^x}{x!}$

- Good when $n \ge 20; p \le 0.05$ or $n \ge 100; np \le 10$.

Continuous Distributions

Continuous uniform distribution: $X \sim U(a, b)$.

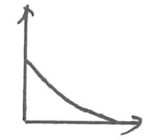
$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

$\mathbb{E}[X] = \frac{a+b}{2} \quad V(X) = \frac{(b-a)^2}{12}$.

c.d.f.: $F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b. \end{cases}$

Exponential distribution: $X \sim \mathrm{Exp}(\lambda)$.

$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$



$\mathbb{E}[X] = \frac{1}{\lambda}$ $\quad Var[X] = \frac{1}{\lambda^2}$.

c.d.f. $F_X(x) = \begin{cases} 1-e^{-\lambda x}, & x \geq 0. \\ 0, & x < 0. \end{cases}$

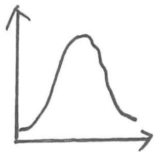
Alternative form: $\mu = \frac{1}{\lambda}$ $\quad f_X(x) = \begin{cases} \frac{1}{\mu} e^{-\frac{x}{\mu}}, & x \geq 0 \\ 0, & x < 0. \end{cases}$

Theorem. $P(X > s+t \mid X > s) = P(X > t)$ ("memoryless")

Normal distribution: $X \sim \mathcal{N}(\mu, \sigma^2)$.

$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)}, \; -\infty < x < \infty$

$\mathbb{E}[X] = \mu$, $Var[X] = \sigma^2$.



Let $Z = \frac{X-\mu}{\sigma}$. Then $Z \sim N(0,1)$

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.

$P(Z \geq 0) = P(Z \leq 0) = \Phi(0) = 0.5$

$\Phi(z) = P(Z \leq z) = P(Z \geq -z) = 1 - \Phi(-z)$.

If $Z \sim N(0,1)$, then $-Z \sim N(0,1)$

If $Z \sim N(0,1)$, then $\sigma Z + \mu \sim N(\mu, \sigma^2)$.

α-th upper quantile: $P(X \geq x_\alpha) = \alpha$.

z upper quantile: $P(Z \geq z_\alpha) = \alpha$.

- $z_{0.05} = 1.645$; $z_{0.01} = 2.326$.

Normal approximation to binomial:

Let $X \sim Bin(n,p)$, s.t. $\mathbb{E}[X] = np$, $V(X) = np(1-p)$.

As $n \to \infty$, $Z = \frac{X - \mathbb{E}[X]}{\sqrt{V(X)}} = \frac{X-np}{\sqrt{np(1-p)}} \approx \sim N(0,1)$.

Continuity correction:

$P(X=k) \approx P(k-\frac{1}{2} < X < k+\frac{1}{2})$

$P(a \leq X \leq b) \approx P(a-\frac{1}{2} < X < b+\frac{1}{2})$

$P(a < X \leq b) \approx P(a+\frac{1}{2} < X < b+\frac{1}{2})$

$P(a \leq X < b) \approx P(a-\frac{1}{2} < X < b-\frac{1}{2})$

$P(a < X < b) \approx P(a+\frac{1}{2} < X < b-\frac{1}{2})$

$P(X \leq c) = P(0 \leq X \leq c) \approx P(-\frac{1}{2} \leq X < c+\frac{1}{2})$

$P(X > c) = P(c < X \leq n) \approx P(c+\frac{1}{2} < X < n+\frac{1}{2})$

Appendix: Table of Normal Distribution

z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
															ADD				
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	36
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517	4	7	11	15	19	22	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224	3	7	10	14	17	20	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	7	10	13	16	19	23	26	29
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	5	8	11	14	16	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	19	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830	2	4	6	8	10	12	14	16	18
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015	2	4	6	7	9	11	13	15	17
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	4	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0