University Integration and Differential Equation

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Trigonometric Identities

- sin, cos: $\sin^2 x + \cos^2 x = 1$
- tan: $\tan x = \frac{\sin x}{\cos x}$
- sec, csc: $\sec x = \frac{1}{\cos x}$; $\csc x = \frac{1}{\sin x}$;
- cot: $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$
- $\sec^2 x - \tan^2 x = 1$; $\csc^2 x - \cot^2 x = 1$
- $\sin(x+y) = \sin x \cos y + \sin y \cos x$
- $\sin 2x = 2 \sin x \cos x$
- $\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = \cos^2 x - 1$
- $\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$
- $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$
- $\tan \frac{x}{2} = \pm\sqrt{(1-\cos x)(1+\cos x)}$
- $\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\sin x \sin y = \frac{\cos(x+y) - \cos(x-y)}{2}$
- $\cos x + \cos y = 2\cos \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\cos x \cos y = \frac{\cos(x-y) + \cos(x+y)}{2}$
- $\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2}$
- sinh, cosh: $\cosh^2 x - \sinh^2 x = 1$
  $\sinh x = \frac{e^x - e^{-x}}{2}$; $\cosh x = \frac{e^x + e^{-x}}{2}$
- tanh: $\tanh x = \frac{\sinh x}{\cosh x}$
- sech, csch: $\operatorname{sech} x = \frac{1}{\cosh x}$; $\operatorname{csch} x = \frac{1}{\sinh x}$
- coth: $\coth x = \frac{1}{\tanh x}$
- $\tanh^2 x + \operatorname{sech}^2 x = 1$; $\coth^2 x - \operatorname{csch}^2 x = 1$
- $\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$
- $\sinh 2x = 2 \sinh x \cosh x$
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
- $\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}$

Common Integrals

Basic
- $\int k\,dx = kx + C$
- $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$
- $\int \frac{1}{x}\,dx = \ln|x| + C$
- $\int e^x\,dx = e^x + C$

Fractional
- $\int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b| + C$
- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\operatorname{sech}^{-1} x + C$
- $\int \frac{1}{|x|\sqrt{1+x^2}}\,dx = -\operatorname{csch}^{-1} x + C$

Logarithmic
- $\int \ln x\,dx = x\ln x - x + C$

Trigonometric
- $\int \cos x\,dx = \sin x + C$
- $\int \sin x\,dx = -\cos x + C$
- $\int \tan x\,dx = \ln|\sec x| + C$
- $\int \sec x\,dx = \ln|\sec u + \tan u| + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \sec x \tan x\,dx = \sec x + C$
- $\int \csc x \cot x\,dx = -\csc x + C$
- $\int \csc^2 x\,dx = -\cot x + C$
- $\int \sinh x\,dx = \cosh x + C$
- $\int \cosh x\,dx = \sinh x + C$
- $\int \operatorname{sech}^2 x\,dx = \tanh x + C$
- $\int \operatorname{csch}^2 x\,dx = -\coth x + C$
- $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C$
- $\int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C$

Special Integrals

- Partial fractions
- Integration by parts:
  $\int u\,dv = uv - \int v\,du$
- $\int \sin^n x \cos^m x\,dx$:
  Use trigonometric identities to convert it into $\sin^k x \cos x$ or $\cos^k x \sin x$.

Differential Equations

1. $M(x) - N(y)y' = 0$

(*Separable*) Separate the variables x and y and rewrite the equation as $\int M(x)\,dx = \int N(y)\,dy$.

2. $y' + P(x)y = Q(x)$

Multiply both sides by an **integrating factor** $\mu(x) = e^{\int P(x)\,dx}$:

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$

$$\mu(x)y = \int \mu(x)Q(x)\,dx$$

3. $y' + P(x)y = Q(x)y^n$

(*Bernoulli*) Let $z = y^{1-n}$, then $z' = (1-n)y^{-n}y'$. Hence we have:

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$
$$\frac{z'}{1-n} + P(x)z = Q(x)$$

and use integrating factor.

4. $ay'' + by' + cy = 0$

Consider the **characteristic equation** $ax^2 + bx + c = 0$ with roots λ_1 and λ_2:
- If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, then $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.
- If $\lambda_1 = \lambda_2 \in \mathbb{R}$, then $y = (c_1 + c_2 x)e^{\lambda x}$.
- If $\lambda_1 \neq \lambda_2 = \alpha \pm \beta i \in \mathbb{C}$, then $y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$.

5. $ay'' + by' + cy = r(x), r(x) \neq 0$

The goal is to find the **particular solution** y_p:
- If $r(x)$ is a polynomial of order n, guess $y_p(x)$ to be a n-th order polynomial.
- If $r(x)$ is in the form of $g(x)e^{kx}$, let $y_p(x) = u(x)e^{kx}$.
- If $r(x)$ is in the form of $g(x)\cos kx$ or $u(x)\sin kx$, let $z(x) = u(x)e^{ikx}$ and take $Re(z)$ or $Im(z)$).

Other Useful Formulae

- (*Fundamental Theorem of Calculus*) $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$
- (*Binomial Expansion*) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$